Spectral Geometry in Non-Standard Domains

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Overview

Background: Heat Equation

Laplacian in $\mathbb{R}^n$: $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$.

- Let $M \in \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary. The heat equation

$$\begin{cases} \frac{\partial \varphi}{\partial t} = -\Delta \varphi \\ \varphi(x,0) = \varphi_0(x) \\ \varphi(x,t) = 0, \ \forall x \in \partial M, t \geq 0. \end{cases}$$

  is the evolution equation for distribution of temperature on M given the initial $(t = 0)$ distribution by φ_0.

- It has a formal solution given by

$$\varphi(x,t) = e^{-t\Delta}\varphi_0, \qquad t > 0$$

Background: Spectrum

An eigenvalue problem for a bounded domain $M \subset \mathbb{R}^n$ with piecewise smooth boundary:

$$\begin{cases} \Delta u = \lambda u \\ u|_{\partial M} = 0, \; u \neq 0. \end{cases}$$

Which gives us a discrete set of positive numbers

$$Spec(M) = \{\lambda_1, \lambda_2, \cdots\}$$

where

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

Background: Weyl's Law

For a bounded domain M with piecewise smooth boundary in $\mathbb{R}^n$

$$N(\lambda) \sim \frac{\omega_n \mathrm{Vol}(M)}{(2\pi)^n} \lambda^{n/2} \qquad \lambda \to \infty$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and

$$N(\lambda) = \#\{\lambda_i \leq \lambda\}$$

is the eigenvalue counting function: total number of eigenvalues less than or equal to a given λ.

Background: Weyl's Law Example

Example

Consider the case of M as *a unit square* in $\mathbb{R}^2$. We have

$$\omega_2 = \pi \times 1^2 = \pi$$

$$Vol(M) = 1 \times 1 = 1$$

Then,

$$N(\lambda) \sim \frac{\pi}{(2\pi)^2} \lambda$$

and this gives us

$$N(\lambda) \sim \frac{1}{4\pi} \lambda, \quad \lambda \to \infty$$

Background: Weyl's Law

Figure: Weyl's Law: One can hear the area of a drum.

Note: By Weyl's Law, one can only hear the area of a drum but not the shape.

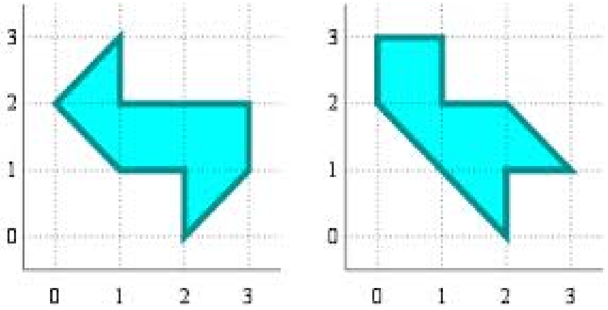

Figure: Isospectral but not isometric

Background: Heat Kernel

- Fundamental solution of the heat equation

$$\begin{cases} \frac{\partial K(t,x,y)}{\partial t} = -\Delta K(t,x,y) \\ K(t,x,y) = 0, \ \forall x,y \in \partial M, t \geq 0. \\ \lim_{t\to 0} K(t,x,y) = \delta_x(y) \ \text{ for all } x,y \in M, \end{cases}$$

- Kernel of the integral operator $e^{-t\Delta}$

$$e^{-t\Delta} f(x) = \int_M K(t,x,y) f(y) dy$$

- In terms of eigenvalues and orthonormal eigenfunctions of Δ

$$K(t,x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).$$

Background: (Asymptotic) Heat Kernel Expansion Theorem

Theorem

Let M be a closed manifold with boundary, the expansion would be of the form

$$K(t,x,x) \sim (4\pi t)^{-\frac{n}{2}}(b_0(x) + b_1(x)t^{\frac{1}{2}} + b_2(x)t + \dots)$$

as $t \to 0$.

In this case, we only know that

$$b_0(x) = 1$$

Background: Jacobi's Theta Function

The Θ-function is

$$\Theta(t) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}, \ t > 0.$$

Using Poisson's summation formula we get the relation:

$$\Theta(t) = \frac{1}{\sqrt{t}} \Theta(\frac{1}{t}), \ t > 0.$$

We also note the useful property that:

$$\Theta(t) \sim \frac{1}{\sqrt{t}}, \qquad t \to 0$$

Background: (Asymptotic) Heat Kernel Expansion Theorem

Theorem

Let M be a closed manifold, with no boundary, then we have an expansion of the form

$$K(t,x,x) \sim (4\pi t)^{-\frac{n}{2}}(a_0(x) + a_1(x)t + a_2(x)t^2 + \ldots)$$

as $t \to 0$.

Additionally we have that

$$a_0(x) = 1$$

$$a_1(x) = \frac{1}{6}S(x)$$

where $S(x)$ is the scalar curvature of M and $S(x) = 2K(x)$ with K(x) denoting the Gaussian curvature.

Background: (Asymptotic) Heat Kernel Expansion Theorem

Example

- Gaussian curvature of a sphere with radius R

$$K(x) = \frac{1}{R^2}$$

- Gaussian curvature of a flat plane

$$K(x) = 0$$

- Gaussian curvature of a cylinder (two-dimensional)

$$K(x) = 0$$

A two-dimensional cylinder comes from a flat plane.

Background: Cases to Consider

We focus on the no boundary cases, thus, we would use the heat kernel expansion for domains with no boundary.

- Sphere: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$
- Fractals
- Quantum Sphere

Cases to Consider: Fractals Example: Cantor Set

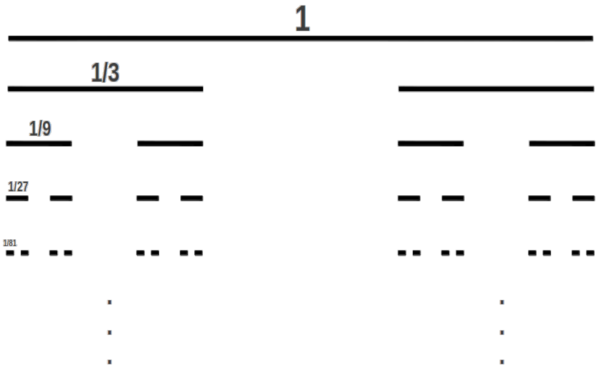

Figure: Construction of the triadic Cantor set. In each iteration, the middle third of all the intervals is removed.

Cases to Consider: Fractals Example: Diamond fractals

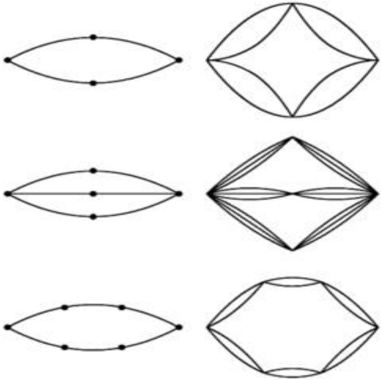

Figure: The first 2 iterations for diamond fractals $D_{4,2}$, $D_{6,2}$, $D_{6,3}$

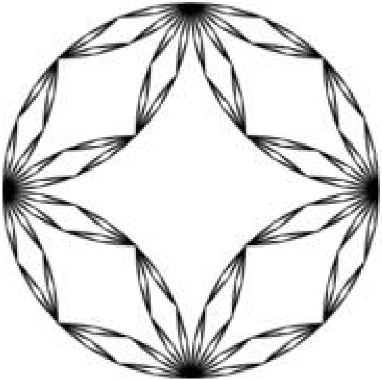

Figure: Further iterations of the diamond fractal $D_{4,2}$

Cases to Consider: Fractals

Hausdorff-Besicovitch dimension d_h:

$$d_h = \lim_{r \to 0} \frac{\ln V(r)}{\ln r}$$

where $V(r)$ is the volume of the fractal at length scale r.

Spectral dimension d_s:

$$d_s = -2 \lim_{t \to 0} \frac{\frac{d}{dt} \ln Z(t)}{\frac{d}{dt} \ln t}$$

where $Z(t)$ is the heat trace.

Walk dimension d_w:

$$d_w = \frac{2d_h}{d_s}$$

Cases to Consider: Quantum Sphere S_q^2 - Noncommutative Geometry

Geometry = Commutative Algebra

Example

Algebra	**Geometry**
$x^2 + y^2 = 1$	circle
$x^2 + y^2 + z^2 = 1$	sphere
$f(x_1, ..., x_n) = 0$	hypersurface

Quantum sphere is noncommutative geometry. It's C^*-algebra.

Cases to Consider: Quantum Sphere S_q^2

Relations for quantum sphere S_q^2
$A, B \in M_n(\mathbb{C})$

$$\begin{aligned} AB &= q^2 BA \\ AB^* &= q^{-2} B^* A \\ BB^* &= q^{-2} A(1-A) \\ B^* B &= A(1-q^2 A) \end{aligned}$$

for some $0 < q < 1$

Eigenvalues and Multiplicities for Laplacian: Sphere S^2

Eigenvalues and multiplicities for Laplacian are

$$\lambda_k = k(k+1),$$
$$deg_k = 2k+1$$

where $k = 0, 1, 2, 3, \ldots$.

Notice that eigenvalues grow **quadratically** with multiplicities growing **linearly**.

Eigenvalues and Multiplicities for Laplacian: Diamond Fractals

Non degenerate eigenvalues: $\pi^2 k^2$, for $k = 1, 2, 3, ...$

Degenerate eigenvalues (iterated eigenvalues):

$$\lambda_k = \pi^2 k^2 L_n^{d_w}$$

obtained by rescaling dimensionless length L_n and time T_n at each iteration n according to $L_n^{d_w} = T_n$. Multiplicities

$$deg_k = BL_n^{d_h} = B(l^{d_h})^n$$

where $B = l^{d_h - 1} - 1$ is the branching factor of the fractal and l^{d_h} is the number of links into which a given link is divided.

The eigenvalues and multiplicities grow **exponentially**.

Eigenvalues and Multiplicities for Dirac operator D: Quantum Sphere

Non degenerate eigenvalues for Dirac operator D such that $D^2 = \Delta$:

$$\lambda_k = \frac{q^{k+1/2} - q^{-(k+1/2)}}{q - q^{-1}}$$

where

$$k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

Note: As $q \to 1$, the above numbers approach to the spectrum of Dirac operator D for the sphere S^2

Heat Trace: Introduction

Definition

$$Z(t) := tr(e^{-t\Delta}) = \sum_{n=1}^{\infty} e^{-\lambda_n t}$$

$$Z(t) = tr(e^{-t\Delta}) = \int_M K(t,x,x)\,dx$$

$$\sim (4\pi t)^{-n/2}\left(\int_M a_0(x) + \int_M a_1(x)t + \int_M a_2(x)t^2 + \ldots\right)$$

in the case *with no boundary*.

Since $a_0 = 1$, the first term in this expansion is the volume of M.

Heat Trace: Sphere S^2

- The heat trace for S^2

$$Z(t) = \sum_{k=0}^{\infty} (2k+1) e^{-(k^2+k)t}.$$

- We will use this and the Euler-Maclaurin Summation Formula to find the expansion for the heat trace.

Heat Trace: Sphere S^2 Details

Use **Euler-Maclaurin Summation Formula**:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{k=2}^{m} \frac{B_k}{k!}\left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) + R_m$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_{2i+1} = 0, i = 1, 2, \ldots$ are the **Bernoulli numbers** and R_m is the remainder.

In our case, $f(k) = (2k+1)e^{-(k^2+k)t}$, $a = 0$, $b = \infty$. So

$$Z(t) = \int_0^\infty (2x+1)e^{-(x^2+x)t}\,dx + \frac{1}{2} +$$

$$+ \frac{1}{6}\left(2e^{-(x^2+x)t} - t(2x+1)^2 e^{-(x^2+x)t}\right)\Big|_0^\infty + \ldots$$

Heat Trace: Sphere S^2 Details Continue

The first integral is:

$$\int_0^\infty (2x+1)e^{-(x^2+x)t}\,dx = -\frac{1}{t}e^{-(x^2+x)t}\Big|_0^\infty = \frac{1}{t}.$$

Then we have

$$Z(t) = \frac{1}{t} + \frac{1}{2} + \frac{1}{12}(-2+t) + \ldots =$$
$$= \frac{1}{4\pi t}\left(4\pi + \frac{4}{3}\pi t + \ldots\right)$$

At the same time, we know that

$$a_0 = Area\left(S^2\right) = 4\pi$$

and

$$a_1 = \frac{1}{3}\int_{S^2} K\,dx = \frac{4}{3}\pi$$

where K is Gaussian curvature and scalar curvature $S(x) = 2K$.
In this way, we can compute all terms in the expansion.

Heat Trace: Diamond fractals

- The heat trace for diamond fractal is:

$$Z(t) = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} + (l^{d_h - 1} - 1) \sum_{n=0}^{\infty} l^{n\, d_h} \sum_{k=1}^{\infty} e^{-k^2\pi^2 t l^{n\, d\omega}}.$$

- Log periodic oscillations of the heat trace

$$Z(t) \sim \frac{c}{t^{\frac{d_s}{2}}} \left(1 + \alpha \cos\left(\frac{2\pi}{d_\omega \log l} \log t + \varphi\right) + \dots\right)$$

for some real constants $c, \alpha$and φ. Recall that

$$d_\omega = \frac{2d_h}{d_s}$$

where d_s is spectral dimension and d_h is Hausdorff dimension.

Heat Trace: Diamond Fractals Graph

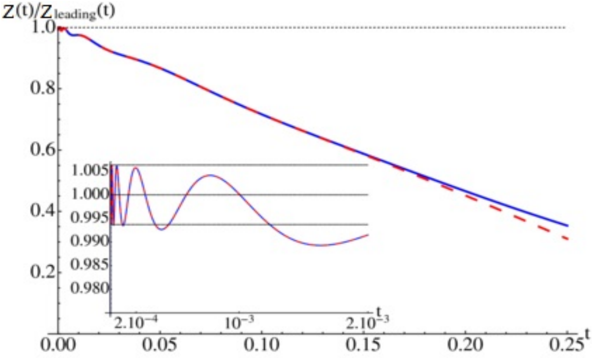

Figure: The log periodic oscillations, at small t, for the heat trace $Z(t)$ on a fractal, normalized relative to the leading term $Z_{leading}(t) = c/t^{d_s/2}$. The solid curve is exact; the dashed curve is the first two terms in the approximate expression.

Heat Trace: Quantum sphere

- The heat trace for quantum sphere is:

$$Z(t) = \sum_{k=1}^{\infty} e^{\frac{q^{-k} - q^{k}}{q - q^{-1}} t}$$

However, we don't know the heat trace expansion for quantum sphere.

Definition

Spectral ζ-function

$$\zeta(s) = \frac{1}{\lambda_1^s} + \frac{1}{\lambda_2^s} + \ldots = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}$$

where $Spec(\Delta) = \{0 < \lambda_1 \leq \lambda_2 \leq \ldots\}$.

- $\zeta(s)$ has analytical continuation

  $$\zeta(s) : \mathbb{C} \setminus \{\frac{n}{2} - j\} \to \mathbb{C}$$

  where $dim\, M = n$ and $j = 0, 1, \ldots$.

- $$\zeta_R(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^s}$$

  is Riemann ζ-function for $Re\ s > 1$.

ζ-function: Sphere S^2

- Polynomial growth of eigenvalues and multiplicities k^2 and k respectively.
- The spectral ζ-function for S^2 is

$$\zeta_{S^2}(s) \sim \sum_{k=1}^{\infty} \frac{k}{(k^2)^s} = \zeta_R(2s-1).$$

- ζ-function has simple real poles $\{1-j\}$ where $j = 0, 1, \ldots$, with the largest pole at $s = 1$.

ζ-function: Diamond fractals

- The spectral ζ-function of the fractals is

$$\zeta_D(s) = \frac{\zeta_R(2s)}{\pi^{2s}} l^{d_h - 1} \left(\frac{1 - l^{1 - d_\omega s}}{1 - l^{d_h - d_\omega s}} \right).$$

- It has simple complex poles at

$$s = \frac{d_s}{2} + i \frac{2\pi m}{d_\omega \log l}, \; m \in \mathbb{Z}$$

with a spectral dimension d_s, a walk dimension d_ω, Hausdorff dimension d_h and a spatial-decimation factor l.

ζ-function: Diamond Fractals Graph

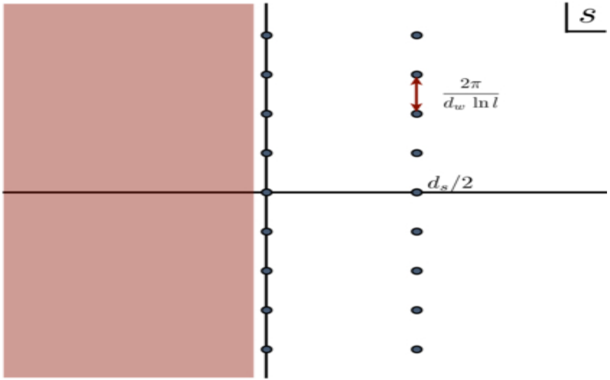

Figure: Sketch of the complex pole structure of the zeta function for a fractal

ζ-function: Quantum Sphere

- The spectral ζ-function of the quantum sphere is

$$\zeta_q(s) = 4(1-q^2)^2 \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{k!\Gamma(s)} \frac{q^{2k}}{(1-q^{s+2k})^2}$$

Note that $\Gamma(s)$ is an extension of factorials to $s \in \mathbb{C}$ with $Re(s) > 0$ and

$$\Gamma(s) = \int_0^{\infty} e^{-t} \frac{t^s}{t} \, dt$$

- All poles of $\zeta_q(s)$ are complex of the second order:

$$-2k + i \frac{2\pi}{\log q} m$$

where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

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Questions