GEOMETRIC MEASURE OF ARENS IRREGULARITY

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Abstract. This report represents the culmination of our group's progress in the Fields Undergraduate Summer Program 2014. Our goal is to study Arens Irregularity by introducing a new, yet natural, measure of Arens Irregularity which yields a number dependent on the irregularity of a Banach Algebra instead of a label. We calculate this new measure, which we call our 'Geometric Invariant' for some Banach Algebras, notably finding the result $\mathfrak{G}(l_1(G)) = 2$ for all discrete groups G, like the Tarski group. We tinker with the definition for our new invariant to eliminate trivial differences between isomorphic algebras, and also investigate some of the structures we rely upon to prove our main results.

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1. Introduction

The aim of this writing is to study some properties of Banach Algebras and its second duals. It is well known that there is a canonical embedding of every Banach space A into A^{**} and that the second dual is often a much larger space than the original space. It is easy to see that one can turn A^{**} into a Banach Space with the operator norm. The next natural question we ask is, if we start with a Banach Algebra B can we define a canonical product on B^{**} so that it becomes a Banach Algebra itself? The answer turns out to be positive and it is not unique. This lack of uniqueness is the central topic of discussion here. We will study two ways in which we can extend the operation from B to all B^{**}, the right and left Arens products. Examples are known for which these operations are not equal and we wish to measure how different they are. There are several ways to do so; here, we introduce a couple of new geometric invariant measures of this disagreement which happen to be positive real numbers.

In previous works, the techniques and definitions developed for calculating this irregularity measures were rather analytical. Here, we develop combinatorial tools that will allow us to calculate this same geometric invariant measure. We will illustrate the connection between discrete notions and notions of interest in the study of Banach Algebras.

In chapter 2 we introduce some basic notions we will need in the future such as the definition of Arens products, Arens regularity and operations on ultrafilters. We also describe an embedding of the set of all ultrafilters on a discrete group G with this operations into the second dual of the Banach Algebra $l_1(G)^{**}$ with each Arens product. [See the paragraph after definition 2.4.] This is the second embedding we will often use since ultrafilters allow us to perform calculations more easily. At the same time, we make sure this calculations hold in the bidual of $l_1(G)$.

In chapter 3 we explore groups and disregard any topological notion, i.e. we consider only discrete groups. We will define the notion of a set being on fire (OF) (see Definition 3.1). The reward we get in looking at this kind of strange sets is that these sets are exactly those which allow us to distinguish between the different multiplications of ultrafilters (see Theorem 3.3). This characterization immediately tells us that we cannot do better in defining these sets; there is no way around on fire sets. Next, we prove the existence of these objects and show some examples. Afterwards we prove that we can always find an on fire set in any infinite and discrete group. Here we wonder whether we can establish some algebraic structure on on-fire sets. By proving some more properties of on fire sets, we now prove that in some cases on fire sets do not have a group structure. Nevertheless, the general question for looking for some algebraic structure for on fire sets of arbitrary groups, more that the Boolean group discussed here, remains open.

In chapter 4, we take a descriptive set theory approach, trying to see whether or not it is possible to write the definition of OF sets in a simpler way. We will thus attempt to categorize the collection of OF sets in the projective hierarchy on the power set of a countable discrete abelian group. We find that this collection in fact is not Borel, by connecting sets to their associated ”trees” and making statements about complexity using an almost visual approach. This gives hints on the appropriate complexity of your definition of OF sets.

We move on to define our Geometric Invariant, the core proposition of this report, in chapter 5. We also move on to introduce two of our main results, 5.3 and 5.4, which evaluate the Geometric Invariant at $l_1(G)$ for countable and infinite discrete groups (respectively). Notably, our result holds for the Tarski group, which had not been studied with respect to Arens Regularity previously.

Our geometric invariant is a substantive new measure of Arens Irregularity, but does admit an important flaw: we can exhibit a situation where isomorphic algebras have different measures, and trivially reach any value in the open interval (0,2). To correct for this, in chapter 6, we introduce a new geometric invariant, 6.5, which is invariant through isomorphisms.

2. Preliminaries

2.1. **Arens Product.** Let $\mathcal{A}$ be a Banach Algebra. We can extend the multiplication operation to the algebra's second dual, $\mathcal{A}^{**}$, in more than one way. These new operations are now presented.

Definition 2.1. [**Arens Product**] For $m, n \in \mathcal{A}^{**}$, $f \in \mathcal{A}^{*}$, and $a, b \in \mathcal{A}$, we have The *Left Arens Product*, $m \square n$,:

$$\langle m \square n, f \rangle = \langle m, n \square f \rangle , \tag{2.1}$$

$$\langle n \square f, a \rangle = \langle n, f \square a \rangle , \tag{2.2}$$

$$\langle f \square a, b \rangle = \langle f, ab \rangle . \tag{2.3}$$

And the *Right Arens Product*, $m \diamond n$,:

$$\langle m \diamond n, f \rangle = \langle n, f \diamond m \rangle , \tag{2.4}$$

$$\langle f \diamond m, a \rangle = \langle m, a \diamond f \rangle , \tag{2.5}$$

$$\langle a \diamond f, b \rangle = \langle f, ba \rangle . \tag{2.6}$$

It is well-known that the two Arens Products do not agree for every Banach Algebra, and thus we classify Banach Algebras with relation to the behavior of their Arens Products. To do so, we will define the left and right topological centers associated to a given Banach Algebra:

$$Z_l\left(\mathcal{A}^{**}\right) := \{X \in \mathcal{A}^{**} \,|\, X \square Y = X \diamond Y \;\forall Y \in \mathcal{A}^{**}\}, \tag{2.7}$$

$$Z_r\left(\mathcal{A}^{**}\right) := \{X \in \mathcal{A}^{**} \,|\, Y \square X = Y \diamond X \;\forall Y \in \mathcal{A}^{**}\}, \tag{2.8}$$

Now, we are able to introduce the mentioned classification.

Definition 2.2 (Classification of Arens Regularity)**.** A Banach Algebra $\mathcal{A}$ is said to be:

(1) **Arens Regular** iff $Z_l\left(\mathcal{A}^{**}\right) = Z_r\left(\mathcal{A}^{**}\right) = \mathcal{A}^{**}$, or equivalently, iff for all $m, n \in \mathcal{A}^{**}$, $m \square n = m \diamond n$.
(2) **Left Strongly Arens Irregular** iff $Z_l\left(\mathcal{A}^{**}\right) = \mathcal{A}$.
(3) **Right Strongly Arens Irregular** iff $Z_r\left(\mathcal{A}^{**}\right) = \mathcal{A}$.
(4) **Strongly Arens Irregular** iff $\mathcal{A}$ is LSAI and RSAI

Equipped with these definitions, many analysts have studied various Banach Algebras and placed them into the AR, LSAI, RSAI, or SAI categories; for example take [1].

However, these measurements do not seem to give a full picture of the underlying structure of the Arens Product; for instance, Algebras exist that are neither SAI nor AR [1]. In general, when measuring "how much these products disagree", naturally, one would like to see a real number. This is the reason why we introduce the **Geometric Arens Irregularity** measure, defined in Sections 3 and 4. In order to be able to calculate this number in several examples, we will make use of ultrafilters. Therefore, it is convenient that we briefly introduce a summary of some useful results we will use ahead.

2.2. **Ultrafilters and their properties.** Since we will work with algebras arising from discrete and infinite groups, we will consider βG, the set of all ultrafilters on the group G.

We'll now introduce some special Banach Algebras that will prove to be important in our discussion, as we can link them to βG in a way made precise below.

Definition 2.3. $l_1(G) = \{f : G \to \mathbb{C} | \sum_{g\in G} |f(g)| < \infty\}$. (All but countably many non-zero values.)
This is a Banach Algebra with convolution given by the formula:

$$fh = \sum_{g\in G} \sum_{ts=g} f(t)h(s)\delta_g.$$

Definition 2.4. $l_\infty(G) = \{f : G \to \mathbb{C} | \sup_{g\in G} |f(g)| < \infty\}$.

Remark 2.5. $l_\infty(G) = l_1(G)^*$, which is an important result that will make calculations much easier for us.

Now, we show we can regard βG as a subspace of the unit ball of $l_1(G)^{**}$. Consider the mapping

$$\psi : \beta G \to l_1(G)^{**}$$

given by $u \mapsto \tilde{u}$, where $\tilde{u}(f) = \int_{g\in G} f(g)du(g)$. (Remember that ultrafilters can be regarded as finitely additive probability measures that only attain the values 0 and 1.) Note that the mapping is well defined, since the integrand is an element of $l_\infty(G) = l_1(G)^*$. Obviously, ψ is linear, so it only remains to show that it is injective, its image is contained in the unit ball and that it is consistent with the first (second) product of ultrafilters and the left (right) Arens product in the second dual. Suppose $\tilde{u} = \tilde{v}$, for some ultrafilters u and v. By considering indicator functions of arbitrary subsets of G, we conclude that the ultrafilters contain exactly the same sets and, hence, $u = v$. Now, lets calculate the norm of $\tilde{u}$, for any $u \in \beta(G)$. We have that

$$||\tilde{u}|| = sup_{||f||=1} | \int_{g\in G} f(g)du(g)| = u(G) = 1,$$

by setting $f = \mathbb{1}_G$. Finally, we must verify that the left (right) Arens product coincides with the left addition (right addition) of ultrafilters in βG, as described in Theorem 2.6, for elements in the power set of G. We now describe the calculation that shows how these operations and ψ are related. Let $u, v \in \beta G$ and $f \in l_\infty(G)$. Then,

$$\begin{aligned}
(\tilde{u}\Box\tilde{v})(f) &\overset{(2.1)}{=} \tilde{u}(\tilde{v}\Box f) \\
&\overset{\text{dfn of } \psi}{=} \int_{x\in G} (\tilde{v}\Box f)(x)du(x) \\
&\overset{(2.2)}{=} \int_{x\in G} \tilde{v}(f\Box x)du(x) \\
&\overset{\text{dfn of } \psi}{=} \int_x \int_y (f\Box x)(y)dv(y)du(x) \\
&\overset{(2.3)}{=} \int_x \int_y f(xy)dv(y)du(x),
\end{aligned}$$

Calculating the resulting product for indicator functions, we retrieve the left product of ultrafilters, as mentioned. Analogously, by use of identities (2.4), (2.5) and (2.6) we can easily check the right product of ultrafilters is recovered from the right Arens product $\diamond$.

This last result will prove itself very useful in the next chapters, as we will use this embedding to calculate the invariants which are of interest to us.

Theorem 2.6. *For any principal ultrafilter $\mathcal{U}$ and any ultrafilter $\mathcal{V}$, we have that $\mathcal{U}\Box\mathcal{V} = \mathcal{U}\diamond\mathcal{V}$.*

Proof. We know that

$$G \hookrightarrow l_1(G) \hookrightarrow l_1^{\star\star}(G).$$

However when embedding $l_1(G) \hookrightarrow l_1^{\star\star}(G)$ it is known that the embedding is in the centre of both Arens products. Moreover we have that this embedding of G into $l_1^{\star\star}(G)$ are the principal ultrafilters. Hence these are in the centre and commute with everything in particular with characteristic functions of sets. □

3. Combinatorial results

Let G be a discrete and infinite group. We will characterize the sets $X \subset G$ such that the function

$$\phi : \beta G \times \beta G \times \mathcal{P}(G) \to \{0, 1, -1\},$$

given by

$$\phi(u, v, X) = u \Box v(X) - u \diamond v(X),$$

is not identically equal to zero. This sets will play a central role in what follows. As usual, here we are thinking of ultrafilters as finitely additive probability measures. Lets start stating the first definition of the section.

Definition 3.1. We say that $X \subset G$ is *on fire* if and only if there are $Y, Y' \subset G$ such that the sets

$$Y' \cap (\cap_{f \in F} X f^{-1}) \neq \emptyset$$

and

$$Y \cap (\cap_{h \in H} h^{-1} X^c) \neq \emptyset$$

for all finite sets $F \subset Y, H \subset Y'$.

Remark 3.2. Observe that here we used the convention,

$$aX := \{ax | x \in X\}$$

and similarly for Xb.

3.1. **Existence of an on fire set.** We now state and prove the relation between on-fire sets and the function ϕ as defined in the first lines of section 3.

Theorem 3.3. *Let G be a group, then $X \subseteq G$ is on fire if and only if there exist a pair of not commutative ultrafilters U and V such that $\phi(U, V, X) \neq 0$.*

Proof. "$\Rightarrow$": Assume $X \subseteq G$ is on fire - there exist $Y, Y' \subseteq G$ satisfying the conditions in the definition 3.1. For addition to be non-commutative, we need to show that $X \in U \Box V$ and $X^c \in U \diamond V$.

As the families $\{Y'\} \bigcup \{Xg^{-1}\}_{g \in Y}$ and $\{Y\} \bigcup \{g'^{-1} X\}_{g' \in Y'}$ both have the finite intersection property they can be extended to a filter, which is then contained in an ultrafilter (as every filter is contained in an ultrafilter). Thus we can find ultrafilters U and V such that they contain $\{Y'\} \bigcup \{Xg^{-1}\}_{g \in Y}$ and $\{Y\} \bigcup \{g'^{-1} X\}_{g' \in Y'}$ respectively.

Expanding the ultrafilter products, we see that

$$\{g \in G \mid Xg^{-1} \in U\} \supseteq Y \in V$$

Thus

$$U \Box V(X) = 1$$

Similarly

$$\{g' \in G \mid g'^{-1} X^c \in V\} \supseteq Y' \in U \Rightarrow U \diamond V(X^c) = 1$$

Therefore U and V are non-commutative; in particular $\phi(U, V)(X) = 1$ and $\phi(U, V)(X^c) = -1$.

"$\Leftarrow$":

There exist some non-commutative ultrafilters U and V such that $\phi(U,V)(X) \neq 0$. Without loss of generality, we can say $X \in U \Box V$ and $X^c \in U \diamond V$, by renaming, if necessary. This is equivalent to the statement

$$U \Box V(X) = 1 \Rightarrow \{g \in G \mid Xg^{-1} \in U\} \in V$$

.

The inner condition, $\{g \in G \mid Xg^{-1} \in U\}$, must yield some infinite set M, as principal ultrafilters are contained in the center of βG, and only principal ultrafilters contain finite sets. So $M \in V$ for this addition to hold. On the other hand,

$$U \diamond V(X^c) = 1 \Rightarrow \{g' \in G \mid g'^{-1}X^c \in V\} \in U$$

The inner condition, $\{g' \in G \mid g^{'-1}X^c \in V\}$, must yield some infinite set M', for the same reason we concluded M is infinite, i.e. in particular non-empty. So $M' \in U$ for this addition to hold.

Then by definition of an ultrafilter, if $F \subset S$ and $G \subset S'$ are finite subsets of the respective infinite sets,

The set $M' \bigcap \left\{ \bigcap_{g \in F} (Xg^{-1}) \right\}$ is infinite, and the set $M \bigcap \left\{ \bigcap_{g' \in G} (g'^{-1}X^c) \right\}$ is also infinite.

Finally, we conclude that X is on fire by setting $Y = M$ and $Y' = M'$. □

Remark 3.4. Note that above proof shows that Definition 3.1 is equivalent to requiring the given finite intersections not only to be non-empty, but also infinite. The first direction of this equivalency is straightforward, the second follows just by Theorem 3.3 as if a set X is on fire in the previously used definition, then there exist ultrafilters U, V s.t. $\phi(U,V)(X) = 1$, but as part of above proof this implies that there are infinite sets M, M' s.t. $M' \bigcap \left\{ \bigcap_{g \in F} (Xg^{-1}) \right\}$ and $M \bigcap \left\{ \bigcap_{g' \in G} (g'^{-1}X^c) \right\}$ both are infinite. Consequently we shall in the following use both of these definitions as our characterisation of on fire sets.

3.2. **Properties of on fire sets.** Next, we show some results related to the intersection, union, and translates of a set not on fire. These results apply for a discrete, abelian, countable group G, and we consider $X \subset G$.

Proposition 3.5. *X is not on fire if and only if the translates of X, Xc, are not on fire, for all $c \in G$.*

Proof. The conditions for a set X being not on fire are $\forall Y, Y'$ infinite there exist F, H finite subsets of Y, Y' respectively such that

$$Y' \cap (\cap_{f \in F} X f^{-1}) = \emptyset$$
$$\text{or } Y \cap (\cap_{h \in H} h^{-1} X^c) = \emptyset.$$

If we examine $Y' \bigcap (\cap_{f \in F} (Xc^{-1}) f^{-1}) = Y' \bigcap (\cap_{f \in F} X(c^{-1}f^{-1}))$, we find that if we replace $(c^{-1}f^{-1}) = f'^{-1}$, we get $Y' \bigcap (\cap_{f \in F} X f'^{-1}) = \emptyset$ since Y' covers all the f'^{-1} as well. Similarly, by replacing $(h^{-1}c) = h'^{-1}$, we get $Y \bigcap (\cap_{h \in H} h'^{-1} X^c) = \emptyset$ if $Y \cap (\cap_{h \in H} h^{-1} X^c) = \emptyset$, which is equivalent to Xc being not on fire.

□

Proposition 3.6. If $X_1, X_2 \subset G$, X_1, X_2 are not on fire, then $X_1 \cup X_2$ is not on fire.

Proof. Assume that $X_1 \cup X_2$ is on fire. Then, for some ultrafilters U and V the next equation holds

$$|U\Box V(X_1 \cup X_2) - U \diamond V(X_1 \cup X_2)| = 1.$$

Without loss of generality, one can assume that $X_1 \cup X_2 \in U\Box V$ and $(X_1 \cup X_2) \notin U \diamond V$, by renaming the ultrafilters if necessary. Now, recall that if the set $A \cup B$ belongs to an ultrafilter, then either A belongs to the ultrafilter or B does. Thus, since $X_1 \cup X_2 \in U\Box V$, there is an $i \in \{1, 2\}$ such that $X_i \in U\Box V$. If $X_i \in U \diamond V$, then $X_1 \cup X_2 \in U \diamond V$ since $X_i \subset X_1 \cup X_2$, a contradiction. Hence, $X_i \notin U \diamond V$ and therefore, X_i is on fire, contradicting the hypothesis.

□

Proposition 3.7. If X_1 and X_2 are not on fire, then $X_1 \cap X_2$ is not on fire

Proof. Again, we proceed by contradiction. Assume that $X_1 \cap X_2$ is on fire. Then, there exist ultrafilters U and V such that

$$U\Box V(X_1 \cap X_2) - U \diamond V(X_1 \cap X_2) \neq 0.$$

Without loss of generality, we can suppose that $X_1 \cap X_2 \in U\Box V$ and therefore $X_1 \cap X_2 \notin U \diamond V$. Since $X_1 \cap X_2 \subset X_1, X_2$ we conclude that $X_1, X_2 \in U\Box V$. Now, both X_1 and X_2 belong to $U \diamond V$ because otherwise some of them would be on fire, contradicting the initial assumption. Thus, $X_1 \cap X_2 \in U \diamond V$, since the latter is an ultrafilter. The last assertion is a contradiction. Hence, $X_1 \cap X_2$ cannot be on fire.

□

3.3. **Examples.**

Example 3.8. $2\mathbb{Z}$ are not on fire regarded as a subset of the integers.

Proof. By definition of addition of ultrafilters, we have:

$$\mathcal{U}\Box\mathcal{V}(2\mathbb{Z}) = \{n|2\mathbb{Z} - n \in \mathcal{U}\} \in \mathcal{V}$$
$$\mathcal{U} \diamond \mathcal{V}(2\mathbb{Z}) = \{n|2\mathbb{Z} - n \in \mathcal{V}\} \in \mathcal{U}$$

We see that if $2\mathbb{Z} \in \mathcal{U}$, then we have that $\{n|2\mathbb{Z} - n \in \mathcal{U}\} \in \mathcal{V} = 2\mathbb{Z} \in \mathcal{V}$. If $2\mathbb{Z} \in \mathcal{V}$, then $\mathcal{U}\Box\mathcal{V}(n\mathbb{Z}) = 1$. If we now look at diamond product $\mathcal{U} \diamond \mathcal{V}(2\mathbb{Z})$, the translates of $2\mathbb{Z}$ which are contained in $\mathcal{V}$ is $2\mathbb{Z}$, which is contained in $\mathcal{U}$, which means that $\mathcal{U}\Box\mathcal{V}(n\mathbb{Z}) = \mathcal{U} \diamond \mathcal{V}(n\mathbb{Z})$. If $2\mathbb{Z} \notin \mathcal{V}$, then $\mathcal{U}\Box\mathcal{V}(n\mathbb{Z}) = 0$, but then we would have $\{n|2\mathbb{Z} - n \in \mathcal{V}\} = 2\mathbb{Z} - 1$, and we know that $2\mathbb{Z} - 1 \notin \mathcal{U}$, thus $\mathcal{U} \diamond \mathcal{V}(2\mathbb{Z}) = 0 \Rightarrow \mathcal{U}\Box\mathcal{V}(2\mathbb{Z}) = \mathcal{U} \diamond \mathcal{V}(2\mathbb{Z})$. The same reasoning holds if $2\mathbb{Z} - 1 \in \mathcal{U}$. Hence, the evens are not on fire.

□

We now take a look at the Boolean group $\mathbb{B}$, which is defined as follows:

Definition 3.9. The *Boolean group* $\mathbb{B}$ is defined to be

$$\mathbb{B} = ([\mathbb{N}]^{<\aleph_0}, \triangle)$$

where $\triangle$ denotes symmetric difference and $[\mathbb{N}]^{<\aleph_0}$ denotes all finite subsets of $\mathbb{N}$.

Example 3.10. For the Boolean group $\mathbb{B}$, we have that the sets S, where if $a \in S, |a| \in 2\mathbb{Z}$ is not on fire.

Proof. If we look at elements in the Boolean group $\mathbb{B}$, we notice that the parity of its cardinality is preserved through the operations of symmetric difference.
Formally, we have $|a \triangle b| = |a| + |b| - 2 * |a \cap b|$. In other words, $\mathbb{B}/S \simeq \mathbb{Z}_2$, and the same analysis done in the previous example holds here.

□

We will next exhibit two particular examples of set which are on fire.

Example 3.11. Subsets of the integers whose complement and itself contain arbitrarily long intervals are on fire.

Proof. Let $X, X^c \subset \mathbb{Z}$ have arbitrarily long intervals. We now verify that that this set is on fire. Set $Y = Y' = \mathbb{Z}$, and let $F \subset Y$, $H \subset Y'$ be finite sets. Let M be the maximum of F and M' be the maximum of H. Now, there are infinitely intervals, both in X and X^c whose lengths are larger than M and M', respectively. Say, if $x, x+1, ..., x+n$ is an interval in X, with $n > M$, then the first intersection condition in the on fire definition contains $x, x+1, ..., x+n-M$. This is true for infinitely many intervals larger than M in X. Hence, the intersection is infinite. Similarly, the second intersection condition is satisfied. Thus, X is on fire.

□

So far, we have characterized sets which are "non-commutative". We will use this property in the future to calculate our geometric invariant $\mathfrak{G}$ for the algebra $l_1(G)$ with convolution, for all discrete and infinite groups. First, we must stop and show that we can find some on-fire set $X \subset G$ for any infinite G. To do so, we will make use of the following:

Lemma 3.12. *Let G be an infinite and discrete group. Then, for all $n \in \mathbb{N}$ and for every finite $M \subset G$ and any $g_1, g_2, ..., g_n \in G$ we have there exists $g \in G$ such that $gg_i^{-1} \notin M \quad$ (for all $i = 1, 2, ..., n$).*

Proof. We proceed by contradiction. Suppose for every $g \in G$ we have there exists an $i \in \mathbb{N}$ such that $gg_i^{-1} \in M$. Then, there exist an i such that the set

$$L_i := \{g \in G | gg_i^{-1} \in M\}$$

is infinite, since the identity $G = \cup_{i=1,2,...,n} L_i$ would imply G is finite. Now, since M is finite, by the pigeonhole principle there are two elements of G, g and h such that $gg_i^{-1} = hg_i^{-1}$. This equation cannot hold in any group unless $g = h$, a contradiction. The conclusion now follows. □

Before we exhibit an on fire set, observe that the lemma holds from both sides.

Theorem 3.13. *Given any infinite and countable group G, there is an on fire set $X \subset G$.*

Proof. The proof is an algorithm. Inductively, we will build two sets, A and B. After this, we chose a set X such that $A \subset X$ and $B \subset X^c$, and prove it has the desired property. In order to make the construction more accessible, we will describe the first steps of the induction explicitly.

Since G is countable, we can list the elements of G as a sequence $G = (g_i)_{i\in\mathbb{N}}$. We now start the calculations, doing it step by step:

Let $A = B = \varnothing$. Fix the dummy variable $n = 1$.
Let $h_1 \in G$ and declare $h_1 g_1 \in A$.
Let $M_2 = \{h_1 g_1\}$.
By the lemma, there is a $k_1 \in G$ such that $g_1 k_1 \notin M_2$. Declare $g_1 k_1 \in B$.
Rename the set M_2 to be $M_2 \cup \{g_1 k_1\}$.

Increase by one the value of n.
By the lemma, there is $h_2 \in G$ such that $h_2 g_i \notin M_2 \quad (i = 1, 2)$. Declare $h_2 g_i \in A \quad (i = 1, 2)$.
Let $M_3 = M_2 \cup \{h_2 g_i\}_{i=1,2}$.
By the lemma, there is a $k_2 \in G$ such that $g_i k_2 \notin M_3$. $\quad (i = 1, 2)$. Declare $g_i k_2 \in B \quad (i = 1, 2)$.
Rename the set M_3 to be $M_3 \cup \{g_i k_2\}_{i=1,2}$.
Increase by one the value of n. And repeat each step, running $i = 1, 2, ..., n$.

It is clear that, in this fashion, we can achive any desired value $n \in \mathbb{N}$ using induction. After this process is done, we obtain sets A and B with certain well defined properties. Now, G is the disjoint union of three sets, $G = A \cup B \cup C$. Declare $X = A \cup C$, and then $X^c = B$. We assert that such X is on-fire by choosing $Y = Y' = G$. First, we must notice that, by construction, $h_i \neq h_j$, if $i \neq j$. Furthermore, we have that for all $j \in \mathbb{N}$ and all $i \leq j$, $h_j g_i \in X$ and $g_i k_j \in X^c$ holds true and it is equivalent to $h_j \in \cap_{i=1,2,...,j} X g_i^{-1}$ and $k_j \in \cap_{i=i,2,...,j} g_i^{-1} X^c$. Now, let $m \geq 0$ and

$$F = \{i_1 < i_2 < ... < i_m\}.$$

It follows from the construction that $h_{i_m} \in X g_1^{-1}, X g_2^{-1}, ..., X g_{i_m}^{-1}$, and hence $h_{i_m} \in \cap_{i=1,2,...,i_m} X g_i^{-1} \subset \cap_{k=1,2,...,m} X g_{i_k}^{-1}$. Also, for every non-negative integer t, we know that $h_{i_m+t} \in \cap_{i=1,2,...i_m+t} X g_i^{-1} \subset \cap_{k=1,2,...,m} X g_{i_k}^{-1}$. Therefore, the hole sequence $\{h_{i_m+t}\}_{t\in\mathbb{N}}$ is contained in $\cap_{k=1,2,...,m} X g_k^{-1}$. Thus, the latter set is infinite, since the $h_i's$ are all different. Analogously, $\cap_{k=1,2,...,l} g_k^{-1} X^c$ is an infinite set. Finally, by setting $Y = Y' = G$, we conclude that X is on-fire. □

Lemma 3.14. *Let G be an infinite group, then there exists an infinite countable subgroup $H \leq G$.*

Proof. Since G is infinite, it is easy to show that we can find a countable infinite subset $S \subseteq G$. Let $S = \{g_i \in G : i \in \mathbb{N}\}$. Consider now the group $H := \langle g_i : i \in \mathbb{N}\rangle$ generated by the elements of S. □

Making use of the above lemma and applying the previous result to this countable subgroup, we can conclude that every infinite group contains an on-fire set.

Now, we will prove that, in some cases, subgroups are necessarily not on fire sets. To do so, we will prove a couple of lemmas. We will assume we can find on-fire subgroups and arrive at a contradiction.

Proposition 3.15. Let K be a group and $G \leq K$. If G is on fire, then there exist sets Y and Y' contained in G^c such that G is on fire with respect to Y and Y'.

Proof. There exists $Y, Y' \subset K$ such that

$$Y' \cap \{ \cap_{f \in F} Gf^{-1} \} \tag{3.1}$$

$$Y \cap \{ \cap_{h \in H} h^{-1} G^c \}, \tag{3.2}$$

are both infinite sets for all finite sets $F \subset Y$ and $H \subset Y'$. Assume that $Y \cap G \neq \varnothing$. Take $f \in Y \cap G$. Then, condition (3.1) tells us that $Y' \cap G$ is infinite, since $Gf^{-1} = G$. Now, consider $g \in Y' \cap G$. Condition (3.2) implies that $Y \cap G^c$ is infinite, since $g^{-1}G^c \subset G^c$. Now, consider $j \in Y \cap G$ and $t \in Y \cap G^c$. Using condition (3.1), we conclude that

$$Y' \cap Gj^{-1} \cap Gt^{-1} = Y' \cap G \cap A,$$

is infinite, where A is some subset of G^c. This is absurd, since the intersection is in fact empty. Hence, the hypothesis $Y \cap G \neq \varnothing$ cannot hold. Therefore, $Y \subset G^c$. Now, since $Y \subset G^c$, for all $f \in Y$, the cosets Gf^{-1} are all contained in G^c. Thus, we can disregard the points in $Y' \cap G$ without altering condition (3.1). Observe that condition (3.2) holds for all finite subsets of Y'. This implies that, in particular, (3.2) is satisfied by finite subsets of $Y' \cap G$. Hence, G is on fire with respect to Y and $Y' \cap G^c$. □

Lemma 3.16. *Given an abelian group K such that that every element distinct form the identity has order p, a prime number. Then, given $G \leq K$, there exist $H \leq K$ such that $K = G \oplus H$.*

Proof. We will prove the existence of H using Zorn's lemma. Let

$$A_G = \{J \leq K | G \cap J = \{id\}\}.$$

First, note that $\{id\} \in A_G$, so this set is non-empty, and $K \notin A_G$. We are considering A_G as a partially ordered set with the partial order given by contention. Let $C_G = \{J_i\}_{i \in I}$ be a chain in A_G, where I is some non-empty index set. We now show that C_G has an upper bound in A_G. Let $T_G = \cup_{i \in I} J_i$. We claim that $T_G \leq K$. Let $x, y \in T_G$. There exists $j \in I$ such that $x, y \in J_j$. Then $x - y \in J_j$, since J_j is a group. Hence, $x - y \in T_G$. This proves the previous assertion. Also, it is clear that $T_G \cap G = \{id\}$. Hence, by Zorn's lemma, we conclude that there exist a maximal element $H \in A_G$. Now, it remains to prove that this maximal element H is a direct summand of K. On the contrary, suppose there exist $x \in (G+H)^c$ and consider the group $\langle H \cup \{x\} \rangle$. We now show that this subgroup is an element of A_G and obtain a contradiction, since $H < \langle H \cup \{x\} \rangle$. Suppose there is $y \in G \cap \langle H \cup \{x\} \rangle$, $y \neq id$. Then $y = kx + h$, for some $k \in \mathbb{Z}$, and some $h \in H$. Hence $kx = y - h \in G + H$. If $k = 0$, then $h = y \neq id$, contradicting the fact $H \in A_G$. Then $k \neq 0$. Since $k \in \{1, 2, ..., p-1\}$, we conclude that $x \in G + H$, a contradiction. Therefore, in any case, y must be equal to the identity. Thus, $G \cap \langle H \cup \{x\} \rangle = \{id\}$, the desired contradiction. This completes the proof. □

Theorem 3.17. *Let K be an abelian group and let $G \leq K$ such that there exist $H \leq K$ such that $K = G \oplus H$. Then G is not on fire.*

Proof. Suppose there is such $G \leq K$ on fire. Then, according to proposition 3.15 there exist $Y, Y' \subset G^c$ such that conditions 3.1 and 3.2 hold. Since we can write $K = G \oplus H$, for some $H \leq K$. Then, for every $y \in Y$ and $y' \in Y'$ there are unique $g, g' \in G$ and$h, h' \in H$ such that $y = g + h$ and $y' = g' + h'$. It follows that

$$G + (g' + h') = G + h'$$

and

$$G^c + (g + h) = G^c + h,$$

so we can forget about the projection of Y and Y' to G without loss of generality; i.e.$Y, Y' \subset H \subset G^c$. (With the only exception of the identity.)
Now, since Y and Y' are infinite, take two points $h, j \in F \subset Y$, where F is a finite set. Then, we have that (3.2) implies that

$$Y' \cap (G + h) \cap (G + j)$$

is infinite. Hence $G+h = G+j$, or else there exists $t, t' \in G$ such that $t+h = t'+j$. Hence, $t - t' = j - h \in G \cap H = \{id\}$. This implies that $j = h$, a contradiction. Therefore, G is never on fire. □

Corollary 3.18. *No subgroup G of the boolean group $\mathbb{B}$ is on fire.*

Proof. Observe that every non-trivial element of the boolean group has order two. Hence, by lemma 3.16, there exists $H \leq \mathbb{B}$ such that $\mathbb{B} = G \oplus H$. The assertion now follows as a consequence of theorem 3.17. □

4. OF sets and the projective hierarchy

Introduction. In the previous section we have been able to give a characterisation of the sets X for which there exist non-commuting ultrafilters U, V (i.e. $U\square V(X) \neq U \diamond V(X)$), specifically through the notion of "on-fire sets" as given in definition 3.1. It is now natural to ask whether this is a "good" characterisation, as it uses existential quantifiers over the power set of the group.
The projective hierarchy over a Polish space gives an appropriate notion to capture this idea. For countable groups we can in fact find a metric on $\mathcal{P}(G)$ so that it becomes a Polish space with the induced topology. This allows us to classify the sets that are on fire within this hierarchy and gives hints on the quality of our characterisation.
We will in fact prove that the collection of o.f. sets is not Borel in this space for a certain class of groups. So this collection is $\mathbf{\Delta}_1^1$-hard but not $\mathbf{\Delta}_1^1$-complete, meaning it is at least at level $\mathbf{\Pi}_1^1$ or $\mathbf{\Sigma}_1^1$ in the projective hierarchy. In a sense this tells us that our definition is in fact a successful notion as this collection cannot be expressed solely as the countable intersection and union of open and closed sets.
The basics for this topic will be also given in some detail below, for further reference please consult [2].

The projective hierarchy.

Definition 4.1. We say (X, τ) is a Polish space if it is homeomorphic to a complete metric space that has a countable dense subset.

Definition 4.2 (Projective hierarchy)**.** Let (X, τ) be a Polish space, then we define the projective hierarchy over X, consisting of the following projective classes $\mathbf{\Sigma}_n^1, \mathbf{\Pi}_n^1, \mathbf{\Delta}_n^1$, inductively: Firstly let $\mathbf{\Sigma}_0^1 = \tau$, i.e. the open sets, then assume we have defined $\mathbf{\Sigma}_n^1$, and let in general

$$\begin{aligned}
\mathbf{\Pi}_n^1 &:= \sim \mathbf{\Sigma}_n^1 = \{A \subseteq X : X \setminus A \in \mathbf{\Sigma}_n^1\} \\
\mathbf{\Delta}_n^1 &:= \mathbf{\Pi}_n^1 \cap \mathbf{\Sigma}_n^1 \\
\mathbf{\Sigma}_{n+1}^1 &:= \{proj_X(A) : \exists Y \text{ Polish}, A \subseteq X \times Y, A \in \mathbf{\Pi}_n^1(X \times Y)\}
\end{aligned}$$

where $proj_X(A) := \{x \in X : (\exists y \in Y)\langle x, y\rangle \in A\}$.

According to [2], chapter 37 we have

$$\mathbf{\Pi}_n^1 \cup \mathbf{\Sigma}_n^1 \subseteq \mathbf{\Delta}_{n+1}^1$$

And so we obtain the following picture for the projective hierarchy:

$$\begin{array}{ccccccc}
 & \mathbf{\Sigma}_0^1 & & \mathbf{\Sigma}_1^1 & & \mathbf{\Sigma}_2^1 & \\
\mathbf{\Delta}_0^1 & & \mathbf{\Delta}_1^1 & & \mathbf{\Delta}_2^1 & & \cdots \\
 & \mathbf{\Pi}_0^1 & & \mathbf{\Pi}_1^1 & & \mathbf{\Pi}_2^1 &
\end{array}$$

where every class is contained in any class to the right of it. Further note that by [2] we have that $\mathbf{\Delta}_n^1$ is closed under countable union, countable intersection and complements. Thus it immediately follows that $\mathbf{\Delta}_0^1$ are exactly the Borel sets in the space (X, τ).

$\mathcal{P}(G)$ as a Polish space.

Definition 4.3. Let G be a countable group, then we can list its elements as $g_1, g_2, \dots$, thus we can define a metric d on $\mathcal{P}(G)$ by

$$d(X,Y) = \begin{cases} 0 \text{ if } X = Y \\ 2^{-n+1} \text{ for } n = \min\{m : g_m \in X \cup Y \setminus X \cap Y\} \text{ o.w.} \end{cases}$$

It is now easy to see that d defines a complete metric on $\mathcal{P}(G)$. We can also check that the collection of finite subsets of G is both countable and dense, thus $\mathcal{P}(G)$ with the induced topology is a Polish space. Thus we can consider the projective hierarchy over $\mathcal{P}(G)$.

Definition 4.4. Let G be any group, we define $Fr(G)$ as a set to be the collection of all sets that are not on fire, according to definition 3.1, i.e.:

$$\begin{aligned} Fr(G) &:= \{X \subseteq G : (\forall Y, Y' \subseteq G)(\exists F \subseteq Y, H \subseteq Y' \text{ both finite}) \\ &\quad Y \cap \bigcap_{f\in F} Xf^{-1} = \emptyset \ \vee \ Y' \cap \bigcap_{h \in H} h^{-1}X^c = \emptyset\} \\ &= \{X \subseteq G : (\forall U, V \in \beta G) U \Box V(X) = U \diamond V(X)\}. \end{aligned}$$

Remark 4.5. Note that all the subsets of G that are on fire are $\mathcal{P}(G) \setminus Fr(G)$. As $\mathbf{\Delta}^1_1$ is closed under complementation it is sufficient to prove that $F(G) \subseteq \mathcal{P}(G)$ is not Borel.

***$Fr(G)$* is not Borel.** Let G be a countable discrete group, in the following we will prove that if G is abelian and such that $G = G_1 \oplus G_2$, where G_1 and G_2 are both isomorphic to G, then $Fr(G)$ is not Borel.
In order to be able to show this we first need to develop some mathematical ideas:

Notation 4.6. Let $(G, \star)$ be an abelian group, then we write for any $g \in G, H \subseteq G$:

$$H + g := H \star g \text{ and } H - g := H \star g^{-1}$$

Notation 4.7. Given a set S and $n \in \mathbb{N}$ the notation S^n will be used to denote all functions from n to S (recall that $n = \{0, 1, 2, \cdots, n-1\}$. The notation $S^{\leq n}$ will be used to denote $\cup S^j{}_{j \leq n}$ and the notation $S^{<\omega}$ will be used to denote $\cup_{j\in\omega} S^j$. If $s \in S^n$ and $j \leq n$ then $s \restriction j$ is the element of S^j obtained by restricting s to j. If $s \in S^n$ and $x \in S$ then $s \frown x$ denotes the sequence in S^{n+1} such that $s \frown x \restriction n = s$ and $s \frown x(n) = x$. On the other hand, $x \frown s$ denotes the sequence defined by $x \frown s(0) = x$ and $x \frown s(j+1) = s(j)$.

A tree on S is a subset $T \subseteq S^{<\omega}$ such that $t \restriction j \in T$ for all $t \in T$ and $j \in \mathbf{dom}(t)$. If $t \in T$ then $\sigma_T(t) = \{x \in S | t \frown x \in T\}$ and $\Sigma_T(t) = \{t \frown x | x \in \sigma_T(t)\}$. The *leaves* of T are defined to be all $s \in T$ such that $s \frown x \notin T$ for all $x \in S$. Let the set of all leaves of T be denoted by $\Lambda(T)$. A *branch* of T is a function $f : \omega \to S$ such that $f \restriction j \in T$ for all $j \in \omega$.

A tree T will be called *ω-branching* if $\sigma_T(t)$ is infinite for all $t \in T \setminus \Lambda T$. If T is *ω-branching* then a (partial) function ρ_T from a tree T to the ordinals is defined by setting $\rho_T(\text{t}) = 0$ if $t \in \Lambda(T)$ and defining $\rho_T(t) := \sup_{s\in\Sigma_T(t)}(\rho_T(s)+1)$ provided that $\rho_T(s)$ is defined for all $s \in \Sigma_T(t)$. The *rank* of T, $\rho(T)$ is defined to be $\rho_T(\emptyset)$ provided that $\rho_T(\emptyset)$ is defined. For any function $F : X \to Y$ and $A \subseteq X$ denote $\{F(a) | a \in A\}$ by $F[A]$.

Definition 4.8. Given a countable discrete abelian group $(G,*)$ and $X \subseteq G$ define the tree $T_{G,X}$ on $G \times G$ to consist of all sequences $s \in (G \times G)^{<\omega}$ such that, letting $s(n) = (s_0^n, s_1^n)$:

- $s_0^n \in \cap_{j \in n} X + s_1^j$
- $s_1^n \in \cap_{j \in n} (G \setminus X) + s_0^j$

This definition naturally gives rise to a mapping $m : \mathcal{P}(G) \to \mathcal{P}((G \times G)^{<\omega})$, with $X \mapsto T_{G,X}$. We can now consider $\mathcal{P}(G)$ as a Polish space with the induced topology given by the metric from definition 4.3, and can define a similar topology f on the powerset of the countable set $(G \times G)^{<\omega}$, s.t. $\mathcal{P}((G \times G)^{<\omega})$ with the induced topology is a Polish space. We now would the like this mapping m to be a Borel homeomorphism as this would imply that the Borel sets are the same, and thus we can further work with the space $\mathcal{P}((G \times G)^{<\omega})$.

Lemma 4.9. *Let G be as previously a countable, discrete, abelian group. Then, the mapping $m : \mathcal{P}(G) \to \mathcal{P}((G \times G)^{<\omega})$, with $X \mapsto T_{G,X}$ is a continuous, injective function and it identifies the Borel sets of its domain and the Borel sets of its image. Moreover, $m(\mathcal{P}(G))$ is a closed subset of $\mathcal{P}((G \times G)^{<\omega})$ and $m(\mathcal{P}(G))$ is a Polish Space and we can thus speak of its Projective Hierarchy.*

Proof. **(m is injective)**: Let e be the identity in G. Then consider $T := \{t \in T_{G,X} : t \restriction 1 = (e,e)\}$. By definition 4.8 we have $\langle (e,e), (a,b) \rangle \in T$ if and only if

$$a = a - e \in X,$$
$$b = b - e \notin X.$$

This defines X uniquely. Note that m is not surjective in the set of all trees in $G \times G$, since the tree

$$\{[(e,e)], [(e,e),(a,b)], [(e,e),(b,a)]\}$$

has empty preimage for $a \neq b$.

(Continuity):
It suffices to show that if $t \in (G \times G)^{<\omega}$ then $\{X \subseteq G | t \in T_{G,X}\}$ is Borel in $\mathcal{P}(G)$. We will proceed by induction on the length of t. If $t = \emptyset$ then $\{X \subseteq G | t \in T_{G,X}\}$ $= \mathcal{P}(G)$ which is a Borel set. Now let $t = s \frown (x,y)$. By the induction hypothesis $B = \{X \subseteq G | s \in T_{G,X}\}$ is Borel. Note that $t \in T_{G,X}$ if and only if $s \in T_{G,X}$ and the following hold:

$$x \in \bigcap_{k \leq |s|} X + t_1^k \tag{4.1}$$

$$y \in \bigcap_{k \leq |s|} (G \setminus X) + t_0^k \tag{4.2}$$

In other words, $t \in T_{G,X}$ if and only if $X \in B$ and Conditions 4.1 and 4.2 hold. Hence it suffices to show that the set of X satisfying Conditions 4.1 and 4.2 are both Borel. We claim that the set of all X satisfying condition 4.1 is open. To see this, one must prove that if Y is a finite subset of G, then B_Y, the set of all subsets of G containing Y, is open. To prove it, consider $Y \subset W$. Then the open ball with radius 2^{-n-1} centered at W is contained in B_Y, where n is the maximum number such that $g_n \in Y$. Then, making $Y = \{x - t_1(0), x - t_1(1), ..., x - t_1(|s|)\}$, it follows that the set of X satisfying condition 4.1 is open and hence Borel.

A similar statement holds for Condition 4.2. Hence, the intersection of both sets is Borel.

(Borel inverse):
Here it suffices to show that the image of a point set $m(\{g\}) = T_{G,\{g\}}$ is Borel. However, this is clearly true as $\{T_{G,\{g\}}\}$ is closed. This establishes a bijection between the Borel sets of $\mathcal{P}(G)$ and the Borel sets of $m(\mathcal{P}(G))$.

($m(\mathcal{P}(G))$ is closed):
Let T be a limit point of $m(\mathcal{P}(G))$. Then, there are $T_n \in m(\mathcal{P}(G))$ such that $T_n \to T$. From the injectivity of m, it follows that there are unique X_n such that $m(X_n) = T_n$. Now, if the sequence of X_n converges in $\mathcal{P}(G)$ to L, then $m(L) = T$. Consider the set $A = \{T_1, T_2, ...\} \cup \{T\}$. A is closed since its unique limit point ($\mathcal{P}((G \times G)^{<\omega})$) is a Hausdorff space) is contained in A. By the continuity of m, we know that $m^{-1}(A)$ is closed in $\mathcal{P}(G)$. Observe that $m^{-1}(A) = \{X_1, X_2, ...\} \cup B$, where $B \in \{\emptyset, \{L\}\}$, for some $L \in \mathcal{P}(G)$ distinct from all X_n. If $B = \{L\}$, then we are done, $T = m(L)$. If B is empty and since A is closed, then for some natural N, X_N is a limit point of A. Therefore, there is a strictly growing divergent subsequence $X_{i_n} \to X_N$. It follows that

$$m(X_N) = m(lim_n X_{i_n}) = lim_n m(X_{i_n}) = lim_n m(X_n) = T.$$

(Recall that the value of the last limit is independent of the choice of divergent sequence.) Again, T is in the image of m.

($m([G]^{<\omega})$ is dense and countable):
Obviously, this set is countable since m is a bijection onto its image. Now, for every element of $m(\mathcal{P}(G))$, T, there is a unique set X such that $m(X) = T$. Since $\mathcal{P}(G)$ is separable, and the set $[G]^{<\omega}$ is dense in $\mathcal{P}(G)$, X can be approximated by finite sets X_n. Now, by the continuity of m and m^{-1}, we are able to interchange limits with m and affirm that the set $\{m(X_1), m(X_2), ..., m(X)\}$ is closed. As we did in the previous paragraph, all of the following limits exist and are interchangeable with m. This is,

$$m(X) = m(lim_n X_n) = lim_n m(X_n) = lim_n T_{X_n}.$$

Finally,we conclude that $m(\mathcal{P}(G))$ is Polish space and we can thus regard its Projective Hierarchy. □

Remark 4.10. By Lemma 4.9 for $F(G)$ not to be Borel in $\mathcal{P}(G)$ it is sufficient to prove that $m[F(G)]$ is not Borel in the Polish space $\mathcal{P}((G \times G)^{<\omega})$.

Lemma 4.11. *A subset $X \subseteq G$ of a countable, discrete, abelian group G belongs to $Fr(G)$ if and only if $T_{G,X}$ has no branch.*

Proof. Firstly we know that $X \in Fr(G)$ is equivalent to X being not on fire, by Theorem 3.3. Now X is not on fire is equivalent to there exist no $Y, Y' \subseteq G$ such that both of the following intersections are non-empty for any finite subsets

$F \subseteq Y, H \subseteq Y'$:

$$Y' \cap (\cap_{f\in F} X f^{-1})$$
$$Y \cap (\cap_{h\in H} h^{-1} X^c).$$

But this again is equivalent to saying there exists no infinite sequence $s \subseteq G \times G$ such that for $s = (s_0, s_1)$ we have

$$s_0^n \in \cap_{j\in n} X + s_1^j$$
$$s_1^n \in \cap_{j\in n} (G \setminus X) + s_0^j$$

i.e. this is equivalent to saying the tree $T_{G,X}$ does not have a branch. □

Corollary 4.12. *If $X \subseteq G$ and $Y \subseteq G$ and neither $T_{G,X}$ nor $T_{G,Y}$ has an branch, then neither does $T_{G,X\cup Y}$ nor $T_{G,G\setminus X}$.*

Proof. This is a direct consequence of lemma 4, proposition 3.6 and the definition of $Fr(G)$. □

So indeed we ultimately want to show that the set of well-founded trees, that is the trees T for which $\rho(T)$ as given in 4.7 is defined, is not Borel. In order to do so we will use the following theorem.

Theorem 4.13 (The Boundedness Theorem for $\mathbf{\Pi}^1_1$-ranks)**.** *Let X be a Polish space, let $A \subseteq X$ be a $\mathbf{\Pi}^1_1$ set and let $\phi : A \to ORD$ be a regular $\mathbf{\Pi}^1_1$-rank, with $\phi(A) = \alpha$. Then $\alpha \leq$ _1 and A is Borel iff $\alpha <$ _1.*
If $\psi : A \to$ _1 is any $\mathbf{\Pi}^1_1$-rank and $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then $\sup(\{\psi(x) : x \in B\}) < \omega_1$.

Proof. For the proof please see [2], theorem 35.23 on page 288. □

Corollary 4.14. *If $S \subseteq \mathcal{P}((G\times G)^{<\omega})$ is $\mathbf{\Pi}^1_1$ and we have any $\mathbf{\Pi}^1_1$-rank $\psi : S \to \omega_1$, such that for all $\alpha \in \omega_1$ there is some $s \in S$ such that the rank of s exists and is at least α, then S is not $\mathbf{\Sigma}^1_1$ in $\mathcal{P}((G \times G)^{<\omega})$.*

Proof. This is an immediate consequence from theorem 4.13. □

Let us now observe a few properties of $Fr(G)$ and $m[Fr(G)]$.

Lemma 4.15. *If $\psi : G \to G_1 \leq G$ is an isomorphism and $T_{G,X}$ is well-founded then so is $T_{G,\psi[X]}$.*

Proof. Using Lemma (and the O. F. notion) it suffices to show that there do not exist $Y \subseteq G$ and $Y' \subseteq G$ both infinite such that

$$(\forall A \in [Y]^{<\aleph_0}) Y' \cap \bigcap_{y\in A} \psi[X] + y \neq \emptyset, \tag{4.3}$$

$$(\forall B \in [Y']^{<\aleph_0}) Y \cap \bigcap_{y\in B} (G \setminus \psi[X]) + y \neq \emptyset. \tag{4.4}$$

Suppose there would exist such Y, Y', then observe that, since the cosets of G_1 are disjoint and Condition 4.3 holds, it follows that $Y \subseteq G_1 + a$ for some $a \in G$. Therefore, $\bigcap_{y\in A} \psi[X] + y \subseteq G_1 + a$ for all $A \in [Y]^{<\aleph_0}$. Hence, there is no loss of generality in assuming that $Y' \subseteq G_1 + a$. From Condition 4.4 it then follows that the stronger version of Condition 4.3:

$$(\forall A \in [Y]^{<\aleph_0}) : \quad Y \cap \bigcap_{y\in A} (G_1 \setminus \psi[X]) + y \neq \emptyset$$

also holds. Hence $\psi^{-1}[Y-a]$ and $\psi^{-1}[Y'-a]$ witness that X is O.F. contradicting that $T_{G,X}$ is well-founded. □

Lemma 4.16. *If $G = \oplus_{i\in\omega} G_i$ and $X \subseteq \cup_{i\in\omega} G_i$ and $X \cap G_i$ is NOF in G_i then X is NOF in G.*

Proof. If not, let Y and Y^* be infinite sets witnessing that X is OF in G. In other words,

$$\forall F \in [Y]^{<\aleph_0} : Y^* \cap \bigcap_{y\in F} X + y \neq \emptyset, \tag{4.5}$$

$$\forall F \in [Y^*]^{<\aleph_0} : Y \cap \bigcap_{y\in F} (G \setminus X) + y \neq \emptyset. \tag{4.6}$$

For $g \in G$ let $g = \Sigma_{i\in\omega} g_i$ be the unique decomposition such that $g_i \in G_i$. let $\sigma(g) = \{i \in \omega | g_i \neq \emptyset\}$.

Claim 4.16.1. *There is an in infinite $\bar{Y} \subseteq Y$ and $R \in [\omega]^{N_0}$ such that Conditions 4.5 and 4.6 hold for $\bar{Y}$ and Y^* and such that one of the following two alternatives holds:*

(1) *there is $Z \subseteq \omega \setminus R$ such that*
 (a) *$y \restriction R = y' \restriction R$ for all y and y' in $\bar{Y}$,*
 (b) *for each $y \in \bar{Y}$ there is a (possibly not unique) $z(y) \in Z$ such that $\sigma(y) = R \cup \{z(y)\}$.*
(2) *There is $P \subseteq [\omega]^2$ such that*
 (a) *$y \restriction_{R\setminus P} = y' \restriction_{R\setminus P}$ for all y and y' in $\bar{Y}$,*
 (b) *if y and y' belong to $\bar{Y}$, then $\sigma(y - y') \subseteq P$.*

Proof of claim: Begin by observing that if y and y' are in Y then $|\sigma(y-y')| \leq 2$ because, letting $F = \{y, y'\}$ and applying Condition 4.5 yields some $y^* \in Y^*$ and x and x' in X such that $y^* = x + y = x' + y'$. Let $x \in G_{i(x)}$ and $x' \in G_{i(x')}$. If $|\sigma(y - y')| \geq 3$ then it is possible to choose $j \in \sigma(y - y') \setminus \{i(x), i(x')\}$. Then $y(j) = x + y(j) = y^*(j) = x' + y'(j) = y'(j)$ contradicting that $j \in \sigma(y - y')$.

Now choose $\bar{y} \in Y$ and for $a \in [\sigma(\bar{y})]^{\leq 2}$ let $Y_a = \{y \in Y | \sigma(y - \bar{y}) \cap \sigma(\bar{y}) = a\}$. Using the above Lemma there is $a \in [\sigma(\bar{y})]^{\leq 2}$ such that Y_a and Y_* satisfy Conditions 4.5 and 4.6. Three cases need to be considered.
Case One. $a = \emptyset$.
In this case $y \restriction R = \bar{y} \restriction R$ for all $y \in Y_\emptyset$. Then

$$Y_0 \subseteq \{y \in Y_0| \ |y \setminus Y_0| = 1\} \cup \{y \in Y_0| \ |y \setminus Y_0| = 2\} \cup \{\bar{y} \restriction R\}$$

and so it is possible to let $\tilde{Y}$ be one of these three sets and still have Conditions 4.5 and 4.6 satisfied. Obviously $\tilde{Y}$ cannot be $\{\bar{y} \restriction R\}$. Observe that if $\tilde{Y} = \{y \in Y_0| \ |y \setminus Y_0| = 2\}$ then $| \cup \{\sigma(y) \setminus R | y \in \tilde{Y}\}| \leq 3$ and so there is $a \subseteq \cup\{\sigma(y) \setminus R | y \in \tilde{Y}\}$ such that $|a| = 2$ and if $\bar{Y} = \{y \in \tilde{Y} | \sigma(y) \setminus R \subseteq a\}$ then $\bar{Y}$ and Y^* satisfy Condition 4.5 and 0.6. In this case the second alternative of the conclusion is immediate.
If $\tilde{Y} = \{y \in Y_0| \ |y \setminus Y_0| = 1\}$ and $\tilde{Y}$ and Y^* sstisfy Conditions 4.5 and 4.6 then let $Z = \cup\{\sigma(y) \setminus R | y \in \tilde{Y}\}$ and the first alternative of the conclusion is immediate.
Case Two. $|a| = 1$.
In this case there is $j \in R$ such that letting $\tilde{Y} = \{y \in Y | \sigma(y - \bar{y}) \cap R = \{j\}\}$ it

follows that Conditions 4.5 and 4.6 are satisfied. Letting $\bar{R} = R \setminus \{j\}$ reduces this case to the first case.
Case Three. $|a| = 2$.
Letting $\bar{R} = R \setminus a$ reduces this case to the first case.
Now consider the first alternative. If $F \subseteq Y$ and $|\{z(y)|y \in F\}| \geq 3$ and $y^* \in \cap_{y \in F} X + y$ then $\sigma(Y^*) \subseteq R$. To see this supose that $j \in \omega \setminus R$. Let $y \in F$ be such that $z(y) \neq j$. Then $y^* = x + y$ for some $x \in X$ and it follows that $x \in X \cap G_j$. Now let $\bar{y} \in F$ such that $z(\bar{y}) \notin \{j, z(y)\}$. Then $y^* = \bar{x} + \bar{y}$ for some $\bar{x} \in X$. Then $y^*(z(y)) = y(z) \neq \emptyset = (\bar{x} + \bar{y})(z)$.
But now it must be that $y^* \restriction R = y \restriction R$ for every $y \in F$. To see this let $y \in F$. There is then some $x \in X$ such that $y^* = x + y$ and, since $z(y) \notin R$ and $\sigma(y^*) \subseteq R$ it must be that $x \in G_{z(y)}$. Hence $y^* \restriction R = (x + y) \restriction R = y \restriction R$. hence Y^* is not infinite.

Now consider the second alternative. First note that, without loss of generality, in this case $\sigma(y^*) \subseteq R \cup P$ for all $y^* \in Y^*$. To see this note that if $y^* \in Y^*$ then, without loss of generality, it can be assumed that there are distinct y and $\bar{y}$ in Y such that $y^* \in (X + y) \cap (X + \bar{y})$ - in other words, there are x and $\bar{x}$ in X such that $y^* = x + y = \bar{x} + \bar{y}$. Now let $j \in \sigma(y^*) \setminus (R \cup P)$ and $\sigma(\bar{y}) \subseteq R \cup P$ it follows that x and $\bar{x}$ must belong to G_j. Hence

$$\bar{y} \restriction (R \cup P) = (\bar{y} + \bar{x}) \restriction (R \cup P) = y^* \restriction (R \cup P) = (y + x) \restriction (R \cup P) = y \restriction (R \cup P)$$

Contradicting that y and $\bar{y}$ are distinct.
Hence it can be assumed that Y and Y^* are subsets of $\oplus_{i \in R \cup P} G_i$ is OF in $\oplus_{i \in R \cup P} G_i$. However, $X = \cup_{i \in R \cup P} G_i \cap X$ and each $X \cap G_i$ is NOF in G_i. By Lemma 4.15 it follows that each $X \cap G_i$ is NOF in $\oplus_{i \in R \cup P} G_i$. By Corollary 4.12 it follows that X is NOF in $\oplus_{i \in R \cup P} G_i$. □

Finally we developed all ideas necessary to prove that $Fr(G)$ is not Borel.

Theorem 4.17. *Let $(G, +)$ be a countable, discrete, abelian group G such that $G = G_1 \oplus G_2$, where G_1 and G_2 are both isomorphic to G. Then $Fr(G)$ is not Borel, i.e. $Fr(G) \notin \mathbf{\Delta}^1_1$ as a subset of $\mathcal{P}(G)$.*

Proof. Firstly note that by lemma 4.9 it is sufficient to prove that $m[Fr(G)]$ is not Borel as a subset of $\mathcal{P}((G \times G)^{<\omega})$. We will prove this by contradiction. Assume that $m[Fr(G)] \in \mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$ as a subset of $\mathcal{P}((G \times G)^{<\omega})$.
Then $m[Fr(G)]$ is $\mathbf{\Pi}^1_1$, moreover the rank function $\rho : m[Fr(G)] \to \omega_1$ as given in 4.7 is a $\mathbf{\Pi}^1_1$-rank. However if we now are able for each ordinal $\alpha \in \omega_1$ to find a set $X \subseteq G$ s.t. $\rho(T_{G,X})$ exists and is at least α, then $m[Fr(G)] \notin \mathbf{\Sigma}^1_1$ by Corollary 4.14 and we reach a contradiction.
We will proceed using transfinite induction.
Firstly note that for $\alpha = 0$, we have that $T_{G,\emptyset} = G \times G^{\leq 1}$ and hence the rank of $T_{G,\emptyset}$ is $1 \geq \alpha$. Now it remains to show the successor and the limit case, i.e. the following:

(1) If there is $X \subseteq G$ such that the rank of $T_{G,X}$ is α then there is $\tilde{X} \subseteq G$ such that the rank of $T_{G,\tilde{X}}$ is at least $\alpha + 1$.
(2) If there are $X_n \subseteq G$ such that the rank of T_{G,X_n} is α_n and $\alpha_n \in \alpha_{n+1}$ and $\alpha = \lim_{n \to \infty} \alpha_n$ then there is $\sqcup_n X_n \subseteq G$ such that the rank of $T_{G, \sqcup_n X_n}$ is at least α.

Each of these assertions will now be established.
(1): Let $\psi : G \to G_1$ be an isomorphism. Let $X \subseteq G$ be a set such that $T_{G,X}$ has rank α. Now fix $z_1 \in G_2$ s.t. $z_1 \neq 0$ and let $Z = G_2 \setminus \{z_1, 0\}$. Now consider the set

$$\tilde{X} = \psi[X] \cup \bigcup_{z \in Z} (G_1 - z).$$

We have $\tilde{X} \cap G_1 = \psi[X]$ which is NOF in G_1 as ψ is an isomorphism, and $\tilde{X} \cap G_2 = G_2 \setminus \{0, z_1\}$ which is NOF in G_2 as its complement is the finite set $\{0, z_1\}$. Thus by lemma 4.16 the set $\tilde{X}$ is NOF in G and the tree $T_{G,\tilde{X}}$ is well-founded.
Now let $t \in T_{G,X} \cap (G \times G)^n$ and $z_2 \in Z$. Define $\tilde{t} = (z_1, z_2) \frown \bar{t}$ where $\bar{t}(k) = (\psi(t_0^k), \psi(t_1^k))$. Then for any $j \in n$ it follows that $\tilde{t}_0^{j+1} = \psi(t_0^j)$ and that for all $0 \leq j \leq n-1$

$$\begin{aligned}
\tilde{t}_0^{j+1} = \psi(t_0^j) &\in \left(\bigcap_{i \in j} \psi[X] + \psi(t_1^i) \right) \cap G_1 \\
&\subseteq \left(\bigcap_{i \in j} \tilde{X} + \psi(t_1^i) \right) \cap ((G_1 - z_2) + z_2) \\
&\subseteq \left(\bigcap_{i \in j} \tilde{X} + \psi(t_1^i) \right) \cap (\tilde{X} + z_2) \\
\text{i.e.} \quad \tilde{t}_0^{j+1} &\in \bigcap_{i \in (j+1)} \tilde{X} + \tilde{t}_1^i
\end{aligned}$$

and similarly, we have $\tilde{t}_1^{j+1} = \psi(t_1^j)$ and

$$\begin{aligned}
\tilde{t}_1^{j+1} = \psi(t_1^j) &\in \left(\bigcap_{i \in j} \psi[G \setminus X] + \psi(t_0^i) \right) \cap G_1 \\
&\subseteq \left(\bigcap_{i \in j} (G \setminus \tilde{X}) + \psi(t_0^i) \right) \cap ((G_1 - z_1) + z_1) \\
&\subseteq \left(\bigcap_{i \in j} (G \setminus \tilde{X}) + \psi(t_1^i) \right) \cap (G \setminus \tilde{X} + z_1) \\
\text{i.e.} \quad \tilde{t}_0^{j+1} &\in \bigcap_{i \in (j+1)} G \setminus \tilde{X} + \tilde{t}_1^i
\end{aligned}$$

Hence if $T^* = \{(z_1, z_2) \frown \bar{t} | z_1, z_2 \text{ as above and } t \in T_{G,X}\} \subseteq T_{G,\tilde{X}}$ then it follows immediately that $\rho_{T^*}((z_1, z_2) \frown \emptyset)$ is equal to the rank of $T_{G,X}$. Hence the rank of $T_{G,\tilde{X}}$ is strictly greater than the rank of $T_{G,X}$.

(2): To prove the second assertion note that the hypothesis on G actually yields an infinite family $\{G_k\}_{k=1}^{\infty}$ of pairwise disjoint subgroups of G, s.t. $G = \oplus_{i \in \omega} G_i$ and each of them is isomorphic to G. Let $\Psi_i : G \to G_i$ be isomorphisms. Given $X_i \subseteq G$ such that the rank of T_{G,X_i} is α_i and $\alpha_i \in \alpha_{i+1}$ and $\alpha = \lim_{i \to \infty} \alpha_i$ define $\sqcup_{i=1}^{\infty} X_i := \cup_{i=1}^{\infty} \psi_i[X_i]$.

As for each j we have $\sqcup_{i=1}^{\infty} X_i \cap G_j = \psi_j[X_j]$ and as ψ_j are isomorphism this is NOF in G_j, thus by lemma 4.16 $\sqcup_{i=1}^{\infty} X_i$ is NOF in G and the corresponding tree is well-founded.
It is moreover easy to see (similarly to the successor case) that

$$\cup_{i=1}^{\infty} T^*_{G,X_i} \subseteq T_{G,\sqcup_{i=1}^{\infty} X_i}$$

where $T^*_{G,X_i} = \{\psi_i(t) | t \in T_{G,X_i}\}$. Let us show this only for one of the two parts that occur: For a given i, n, let $t \in T_{G,X_i} \cap (G \times G)^n$ and define $\tilde{t} = (\tilde{t}_0^k, \tilde{t}_1^k) := \psi_i(t) = (\psi_i(t_0^k), \psi_i(t_1^k))$. Then for each $j \in n$

$$\begin{aligned} \tilde{t}_1^j = \psi_i(t_1^j) &\in \left(\bigcap_{i \in j} \psi_i[G \setminus X_i] + \psi_i(t_0^i) \right) \\ &\subseteq \left(\bigcap_{i \in j} G \setminus \sqcup_{i=1}^{\infty} X_i + \psi_i(t_0^i) \right) \\ \text{i.e.} \quad \tilde{t}_1^{j+1} &\in \bigcap_{i \in j} G \setminus \sqcup_{i=1}^{\infty} X_i + \tilde{t}_0^i \end{aligned}$$

Thus we find that $\rho(T_{G,\sqcup_{i=1}^{\infty} X_i}) \geq \rho(T_{G,X_i})$ for all i, thus $\rho(T_{G,\sqcup_{i=1}^{\infty} X_i}) \geq \alpha$ as required. □

Remark 4.18. As $\mathbf{\Delta}_1^1$ is closed under complements, this of course implies that also $\mathcal{P}(G) \setminus Fr(G)$ is not Borel.

5. Initial geometric invariant

We now, finally, introduce the Geometric Arens Irregularity measurement, and use the connection between $l_1(G)$ and βG discussed in the introduction to calculate our measurement for all discrete groups.

Definition 5.1. Let $\mathcal{A}$ be a Banach Algebra. We define $\mathfrak{G}_1$, the Geometric Arens Irregularity measurement of $\mathcal{A}$ to be

$$\mathfrak{G}_1(\mathcal{A}) = \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||m \Box n - m \diamond n||$$

where B_1 is the unit ball of $\mathcal{A}^{**}$.

Theorem 5.2 (Properties of the Geometric Invariant)**.** *We see that:*

1. $\mathfrak{G}_1(\mathcal{A})$ *lies in the interval* $[0, 2]$
2. $\mathfrak{G}_1$ *is an isometric invariant*
3. $\mathfrak{G}_1(\mathcal{A}) = 0 \Leftrightarrow \mathcal{A}$ *is Arens Regular*
4. *If* $\mathcal{A}_0$ *is a subalgebra of* $\mathcal{A}$ *then* $\mathfrak{G}_1(\mathcal{A}_0) \leq \mathfrak{G}_1(\mathcal{A})$

Proof. (1): By the properties of a norm clearly $\mathfrak{G}_1(\mathcal{A}) \geq 0$, further by the triangle inequality on the norm $\|\cdot\|$ and as the norm on $\mathcal{A}^{\star\star}$ is submultiplicative we have:

$$\begin{aligned}
\mathfrak{G}_1(\mathcal{A}) &= \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||m \Box n - m \diamond n|| \\
&\leq \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||m \Box n\| + \|m \diamond n|| \\
&\leq \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||m\|\|n\| + \|m\|\|n|| = 2
\end{aligned}$$

(2): Let $\psi : \mathcal{A} \to \mathcal{B}$ be an isometry (i.e. an isometric isomorphism) between two banach algebras. Then we have

$$\begin{aligned}
\psi^\star : \mathcal{A}^\star &\to \mathcal{B}^\star \\
f &\mapsto f \circ \psi^{-1}
\end{aligned}$$

is a canonical isometry, and similarly we find a canonical isometry $\psi^{\star\star} : \mathcal{A}^{\star\star} \to \mathcal{B}^{\star\star}$ that preserves both Arens products. Thus

$$\begin{aligned}
\mathfrak{G}_1(\mathcal{A}) &= \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||m \Box n - m \diamond n||_{\mathcal{A}^{\star\star}} \\
&= \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||\psi^{\star\star -1} \circ \psi^{\star\star}(m \Box n - m \diamond n))||_{\mathcal{A}^{\star\star}} \\
&= \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||\psi^{\star\star}(m \Box n - m \diamond n)||_{\mathcal{B}^{\star\star}} \\
&= \sup_{m,n \in B_1(\mathcal{A}^{\star\star})} ||\psi^{\star\star}(m) \Box \psi^{\star\star}(n) - \psi^{\star\star}(m) \diamond \psi^{\star\star}(n)||_{\mathcal{B}^{\star\star}} \\
&= \sup_{m',n' \in B_1(\mathcal{B}^{\star\star})} ||m' \Box n' - m' \diamond n'||_{\mathcal{B}^{\star\star}} = \mathfrak{G}_1(\mathcal{B})
\end{aligned}$$

(3): We have:

$$\begin{aligned}\mathfrak{G}_1(\mathcal{A}) = 0 &\Leftrightarrow m\Box n = m \diamond n \text{ for all } m, n \in B_1(\mathcal{A}^{\star\star})\\ &\Leftrightarrow \frac{m}{\|m\|}\Box\frac{n}{\|n\|} = \frac{m}{\|m\|} \diamond \frac{n}{\|n\|} \text{ for all } m, n \in \mathcal{A}^{\star\star}\\ &\Leftrightarrow \frac{1}{\|m\|\|n\|}(m\Box n) = \frac{1}{\|m\|\|n\|}(m \diamond n) \text{ for all } m, n \in \mathcal{A}^{\star\star}\\ &\Leftrightarrow m\Box n = m \diamond n \text{ for all } m, n \in \mathcal{A}^{\star\star}\\ &\Leftrightarrow \mathcal{A} \text{ is Arens Regular.}\end{aligned}$$

(4): If $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ then

$$\begin{aligned}\mathfrak{G}_1(\mathcal{A}_0) &= \sup_{m,n\in B_1(\mathcal{A}_0^{\star\star})\subseteq B_1(\mathcal{A}^{\star\star})} ||m\Box n - m \diamond n||\\ &\leq \sup_{m,n\in B_1(\mathcal{A}^{\star\star})} ||m\Box n - m \diamond n|| = \mathfrak{G}_1(\mathcal{A})\end{aligned}$$

□

Theorem 5.3. *Let G be a countable and discrete group. Then $\mathfrak{G}_1(l_1(G)) = 2$.*

Proof. To show this is true, we'll find one such configuration that yields the value two, the maximum possible value, and this will suffice.

Given $m, n \in \beta G$, we know how these elements behave on characteristic functions, and that we can evaluate them at any linear combination of characteristic functions f that still satisfy $||f|| = 1$. We adopt the box/triangle notation for the left and right Arens products, but could have used the dots used above. Choose $f_1(x)$ to be the characteristic function of X, and $f_2(x) = 1 - f_1(x)$, which are both in the unit ball and are just individually the usual characteristic functions.

If we consider the set X on fire, then we know there exist $m, n \in \beta G$ such that

$$\langle m \,\Box\, n - m \diamond n, f_1\rangle = 1$$

and

$$\langle m \,\Box\, n - m \diamond n, f_2\rangle = -1$$

.

Now, considering the function $f(x) = f_1(x) - f_2(x)$

$$\begin{gathered}\langle m \,\Box\, n - m \diamond n, f\rangle =\\ \langle m \,\Box\, n - m \diamond n, f_1\rangle -\\ \langle m \,\Box\, n - m \diamond n, f_2\rangle = 2\end{gathered}$$

We've found the maximum for some specific function, so clearly,

$$\mathfrak{G}_1(l_1(G)) = \sup_{m,n\in\mathfrak{B}_{l_1(G)^{**}}} ||m \,\Box\, n - m \diamond n|| = 2. \tag{5.1}$$

□

Theorem 5.4. $\mathfrak{G}_1(l_1(G)) = 2$, *for G a discrete and infinite group.*

Proof. The countable case follows from 5.3 Since we can find a countable subgroup of G using 3.14, we see that if $A \leq G$, then $l_1(A)$ is a sub-algebra of $l_1(G)$. By the previous section, $\mathfrak{G}_1(l_1(A)) = 2$ and so using properties of our invariant, $\mathfrak{G}_1(l_1(G)) \geq 2$ and since 2 is the maximum possible value, $\mathfrak{G}_1(l_1(G)) = 2$. □

6. A new geometric invariant

Weakness of $\mathfrak{G}_1$. As we could see previously we can find banach algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathfrak{G}_1(\mathcal{A}) = 0$ and $\mathfrak{G}_1(\mathcal{B}) = 2$. Now it is natural to ask the question whether for all $\alpha \in (0, 2)$ there is a banach algebra $\mathcal{C}$ such that $\mathfrak{G}_1(\mathcal{C}) = x$. Indeed for the given geometric invariant there is an easy way of constructing an appropriate banach algebra:
Let A be a set and $s \in A$ following [1] we will write δ_s for the characteristic function of the singleton s and χ_A for the characteristic function of A.

Definition 6.1. Let G be a discrete group and $c \in \mathbb{R}, c \geq 1$, then define

$$l_1(G, c) = \{f : G \to \mathbb{C} : \|f\|_c = \sum_{g \in G} |f(g)|c < \infty, |supp(f)| \leq \aleph_0\}.$$

Definition 6.2. Let $x, y \in G$ we define a product - convolution - on $l_1(G, c)$ by

$$\delta_x \star \delta_y = \delta_{xy}.$$

Note that this defines convolution on the whole space $l_1(G, c)$.

Clearly we have $l_1(G, 1) = l_1(G)$, and one can easily check that for all such constants c, $l_1(G, c)$ together with convolution as defined above and the usual addition form Banach algebras and are moreover isomorphic. Additionally note that as sets $l_1(G, c)$ and their dual spaces are equal for all such c.

Proposition 6.3. Suppose we are given $\alpha \in (0, 2]$, we can find a Banach algebra $\mathcal{A}$ such that

$$\mathfrak{G}_1(\mathcal{A}) = \alpha.$$

Proof. Let G be a discrete and infinite group. Then given $\alpha \in (0, 2]$, let $c = \frac{2}{\alpha} \geq 1$ and consider $l_1(G, c)$. For $f \in l_1^\star(G, c)$ we have the norm

$$\begin{aligned}\|f\|_{c,\star} &= \sup_{x \in l_1(G,c), \|x\|_c = 1} |f(x)| \\ &= \sup_{x \in l_1(G), \|x\| = 1/c} |f(x)| \\ &= \sup_{x \in l_1(G), \|x\| = 1} \frac{1}{c}|f(x)|.\end{aligned}$$

$$\therefore \|f\|_{c,\star} = \frac{1}{c}\|f\|_\star. \tag{6.1}$$

Thus on the second dual space we have the norm, let $m \in l_1^{\star\star}(G, c)$:

$$\begin{aligned}\|m\|_{c,\star\star} &= \sup_{f \in l_1^\star(G,c), \|f\|_{c,\star} = 1} |m(f)| \\ &\overset{(6.1)}{=} \sup_{f \in l_1^\star(G), \|f\|_\star = c} |m(f)| \\ &= \sup_{f \in l_1^\star(G), \|f\|_\star = 1} c|m(f)|.\end{aligned}$$

$$\therefore \|m\|_{c,\star\star} = c\|m\|_{\star\star}. \tag{6.2}$$

Now consider the value of $\mathfrak{G}(l_1(G,c))$, we have:

$$\begin{aligned}\mathfrak{G}(l_1(G,c)) &= \sup_{\|m\|_{c,\star\star}=\|n\|_{c,\star\star}=1} \|m \,\Box\, n - m \,\diamond\, n\|_{c,\star\star} = \\ &\overset{(6.2)}{=} \sup_{\|m\|_{\star\star}=\|n\|_{\star\star}=1/c} c\|m \,\Box\, n - m \,\diamond\, n\|_{\star\star} = \\ &= \frac{1}{c^2} \sup_{\|m\|_{\star\star}=\|n\|_{\star\star}=1} c\|m \,\Box\, n - m \,\diamond\, n\|_{\star\star} \\ \therefore \mathfrak{G}(l_1(G,c)) &= \frac{1}{c}\mathfrak{G}(l_1(G))\end{aligned}$$

By theorem 5.4 we know $\mathfrak{G}(l_1(G)) = 2$. Thus we have

$$\mathfrak{G}(l_1(G,c)) = \alpha.$$

□

This is an example of a collection of algebras that yield any value for the geometric invariant in $(0,2]$. Moreover all of them are isomorphic and [1] shows that $l_1(G)$ is Strongly Arens Irregular (definition 2.2), thus all of them are. However we are interested in establishing a geometric measure of the Arens products, i.e. the algebraic behaviour of the second dual space together with either of the Arens products. As such its value should not alter if we just change the norm of the original Banach algebra and leave the algebraic structure unchanged as in the example above.

New geometric invariant and properties. This motivates the definition of an alternative geometric invariant to measure the difference between the two Arens products on the dual space of an algebra. In the following, when talking about isomorphisms between Banach algebras, we only consider continuous mappings with continuous inverses, i.e.

Definition 6.4. Let $\mathcal{A}, \mathcal{B}$ be a Banach algebras, then we say the algebras are isomorphic, $\mathcal{A} \cong \mathcal{B}$, if there exists a bijective map $\psi : \mathcal{A} \to \mathcal{B}$, s.t. ψ is a homomorphism with regard to the algebraic structure on the algebras, and is continuous with continuous inverse.

Definition 6.5. Let $\mathcal{A}$ be a Banach algebra. We define the new geometric invariant $\mathfrak{G}_2$ to be

$$\mathfrak{G}_2(\mathcal{A}) = \sup_{\mathcal{B}\cong\mathcal{A}} \mathfrak{G}_1(\mathcal{B}) := \sup\{x \in [0,2] : \exists \mathcal{B} \cong \mathcal{A} \text{ s.t. } \mathfrak{G}_1(\mathcal{B}) = x\}$$

Loosely speaking this means we are taking now the value of $\mathfrak{G}_1$ of all algebras $\mathcal{B}$ isomorphic to $\mathcal{A}$ into account.

Remark 6.6. Note that clearly $\mathfrak{G}_2(\mathcal{A}) \in [0,2]$ for any Banach algebra $\mathcal{A}$ since $\mathfrak{G}_1(\mathcal{A}) \in [0,2]$.

Moreover we have for any discrete infinite group G that $\mathfrak{G}_1(l_1(G)) = 2$ thus this implies that $\mathfrak{G}_2(l_1(G,c)) = 2$ for all $c \in \mathbb{R}, c \geq 1$. Hence this geometric invariant is less sensitive to changes solely in the norm of a Banach algebra.

Lemma 6.7. *Let $\mathcal{A}, \mathcal{B}$ be two Banach algebras then*

$$\mathcal{A} \cong \mathcal{B} \Longrightarrow \mathcal{A}^{\star\star} \cong \mathcal{B}^{\star\star}$$

where the isomorphism between $\mathcal{A}^{\star\star}$ and $\mathcal{B}^{\star\star}$ preserves both Arens products.

Proof. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a continuous isomorphism with continuous inverse. Then let $\phi^\star : \mathcal{A}^\star \to \mathcal{B}^\star$ be given by

$$\langle \phi^\star(f), x\rangle = \langle f, \phi^{-1}(x)\rangle$$

Claim: $\phi^\star$ is a continuous bijection with continuous inverse.
Proof of Claim: We have

$$\begin{aligned} &\phi^\star(f_1) = \phi^\star(f_2) \\ \Longrightarrow\ &\langle f_1, \phi^{-1}(x)\rangle = \langle f_2, \phi^{-1}(x)\rangle \ \forall x \in \mathcal{B} \\ \Longrightarrow\ &f_1 = f_2 \end{aligned}$$

Also for any $g \in \mathcal{B}^\star$ we have $f \in \mathcal{A}^\star$ given by $f = g \circ \phi$ is clearly linear and continuous since both g and ϕ are. Thus $f \in \mathcal{A}^\star$ and also $\phi^\star(f) = g$ thus $\phi^\star$ is a bijection.
Moreover we have

$$\begin{aligned} \|\phi^\star(f)\|_{\mathcal{B}^\star} &= \sup_{x\in\mathcal{B}, \|x\|=1} |\langle \phi^\star(f), x\rangle| = \\ &= \sup_{x\in\mathcal{B}, \|x\|=1} |\langle f, \phi^{-1}(x)\rangle| \leq \\ &\overset{\phi \text{ is cts}}{\leq} \sup_{x\in\mathcal{B}, \|x\|=1} |\langle f, x\rangle| K = \\ &= K\|f\|_{\mathcal{A}^\star} \end{aligned}$$

for some constant K. One can analogously show that its inverse is continuous, so $\phi^\star$ is a linear continuous bijection with continuous inverse.

Further consider now $\phi^{\star\star} : \mathcal{A}^{\star\star} \to \mathcal{B}^{\star\star}$ given by

$$\langle \phi^{\star\star}(m), f\rangle = \langle m, \phi^{\star -1}(f)\rangle$$

Similarly to above one can easily check that $\phi^{\star\star}$ is a linear continuous bijection with continuous inverse. Additionally it is straightforward to show that the map $\phi^{\star\star}$ preserves both Arens products, by just using definition 2.1. Hence $\mathcal{A}^{\star\star} \cong \mathcal{B}^{\star\star}$. □

Proposition 6.8. Let $\mathcal{A}$ be a Banach algebra, then

$$\mathfrak{G}_2(\mathcal{A}) = 0 \Longleftrightarrow \mathcal{A} \text{ is Arens Regular.}$$

Proof. By definition 2.2 we have $\mathcal{A}$ is Arens Regular iff:

$$\begin{aligned} &Z_l(\mathcal{A}^{\star\star}) = Z_r(\mathcal{A}^{\star\star}) = \mathcal{A}^{\star\star} \\ \Longleftrightarrow\ &X \,\square\, Y = X \diamond Y \ \forall X, Y \in \mathcal{A}^{\star\star}. \end{aligned}$$

Thus we clearly have

$$\begin{aligned} \mathfrak{G}_2(\mathcal{A}) = 0 &\Longrightarrow X \,\square\, Y - X \diamond Y = 0 \ \forall X, Y \in \mathcal{A}^{\star\star} \\ &\Longrightarrow \mathcal{A} \text{ is Arens regular.} \end{aligned}$$

Moreover if $\mathcal{A}$ is Arens regular, and $\mathcal{B} \cong \mathcal{A}$, then by Lemma 6.7 there exists an isomorphism $\phi : \mathcal{A}^{\star\star} \to \mathcal{B}^{\star\star}$. Thus we have for all $X, Y \in \mathcal{B}^{\star\star}$

$$\begin{aligned} &\phi^{-1}(X) \,\square_{\mathcal{A}} \phi^{-1}(Y) = \phi^{-1}(X) \diamond_{\mathcal{A}} \phi^{-1}(Y) \\ \Longrightarrow\ &\phi^{-1}(X \,\square_{\mathcal{B}} Y) = \phi^{-1}(X \diamond_{\mathcal{B}} Y) \\ \Longrightarrow\ &X \,\square_{\mathcal{B}} Y = X \diamond_{\mathcal{B}} Y. \end{aligned}$$

Thus $\mathcal{B}$ is Arens Regular as well, so $\mathfrak{G}_1(\mathcal{B}) = 0$ for all $\mathcal{B} \cong \mathcal{A}$, so

$$\mathfrak{G}_2(\mathcal{A}) = 0.$$

□

7. Open questions

Question 7.1. Let $\mathcal{A}$ be a Banach Algebra, and $\mathcal{A}_0$ a subalgebra of $\mathcal{A}$, is it true that

$$\mathfrak{G}_2(\mathcal{A}_0) \leq \mathfrak{G}_2(\mathcal{A})?$$

Question 7.2. Does there exist a Banach Algebra $\mathcal{A}$ such that

$$\mathfrak{G}_2(\mathcal{A}) \in (0, 2)?$$

Question 7.3. Can we in fact tell anything about $\mathfrak{G}_1(\mathcal{A})$ when $\mathcal{A}$ is a Strongly Arens Irregular Banach algebra? What about the contrary, does $\mathfrak{G}_1(\mathcal{A}) = 2$ or > 0 tell us whether or not $\mathcal{A}$ is Strongly Arens Irregular?

Question 7.4. Are there O.F. sets with algebraic structure, such as subgroups? (See Corollary 3.18 for subgroups of the Boolean group.)

Question 7.5. Let G be a locally compact group, is it true that $\mathfrak{G}_2(L_1(G)) = 2$? i.e. Can we extend our result past the discrete case to locally compact groups?

Question 7.6. Is it true that on fire. sets are Σ_1^1-complete in any group G?

Question 7.7. Let $\mathbb{T}$ be the unit circle as usually, what is $\mathfrak{G}_2(L_1(\mathbb{T}))$?

Question 7.8. For any discrete group G, $WAP(l_1(G)) \leq l_\infty(G)$ is the algebra of $A \subset G$ such that for all U and $V \in l_\infty^*(G)$ we have that $U \diamond V(A) = U \Box V(A)$. We also define $Fr(G)$ to be the subalgebra of $l_1^*(G) = l_\infty(G)$ of all freeze or not on fire sets of G. The question is to know weather $WAP(l_1(G)) = Fr(G)$ or $WAP(l_1(G)) \subsetneq Fr(G)$?

Question 7.9. This question relates to the result theorem 4.17 in section 4:
Let G be a countable discrete abelian group, whose power set is turned into a Polish space as given via definition 4.3. Is the collection of subsets $Fr(G)$ a $\mathbf{\Pi}_1^1$ set in this space? (Note that this would by theorem 4.17 imply that it is in fact $\mathbf{\Pi}_1^1$-complete.) If it is not, what can be said about its position in the projective hierarchy on the space $\mathcal{P}(G)$?

Question 7.10. With reference to section 4 it is worth noting that if $S = (N, *)$ is the semigroup where the operation $*$ is given by $a * b = max(a, b)$ then $Fr(S) = \mathcal{P}(N)$ and hence is not Π_1^1-complete.
It thus appears to be interesting to characterize the semigroups S (or groups) for which $Fr(S)$ is Π_1^1-complete.
Semigroups for which the answer does not seem to be known include: $(Z, +)$, (N, x), $(Q, +)$, $\Pi_{j=1}^{\infty} Z_{n_j}$ for most sequences $\{n_j\}_{j=1}^{\infty} \subseteq N$ and $(N, *)$ where $a * b$ is the least common multiple of a and b.

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