# Path categories and algorithms

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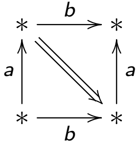
# Geometric concurrency

**Basic idea**: represent the simultaneous execution of processors $a$ and $b$ as a picture (2-cell)




Simultaneous action of multiple processors is represented by higher dimensional cubes.

Restrictions on the system arising from shared resources are represented by removing cubical cells of varying dimensions, so one is left with a cubical subcomplex $K \subset \square^N$ of an $N$-cell, where $N$ is the number of processors.

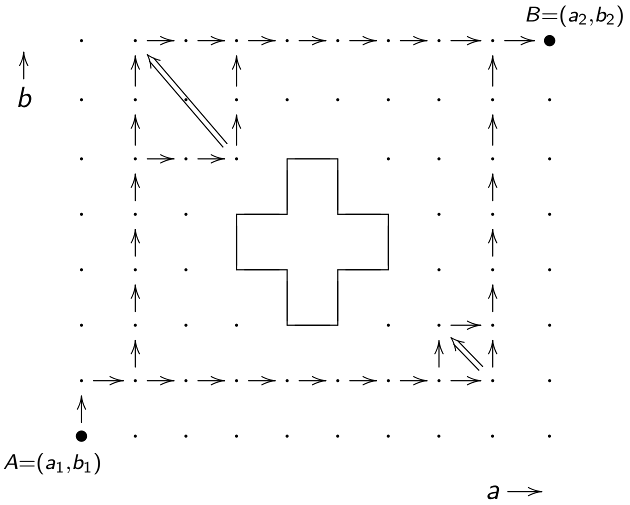
# Higher dimensional automata

*Higher dimensional automata* are cubical subcomplexes $K \subset \square^N$ of the "standard" $N$-cell $\square^N$ (Pratt, 1991).

States are objects (vertices), and "execution paths" are morphisms of the "path category" $P(K)$.

Execution paths are equivalence classes of combinatorial paths through the complex. Executions paths between states $x$ and $y$ are the morphisms $P(K)(x, y)$.

**Basic problem**: Compute $P(K)(x, y)$.


B=(a₂,b₂)
b
A=(a₁,b₁)
a

# Cubical homotopy theory

The $n$-**cell** $\square^n$ is the poset

$$\square^n = \mathcal{P}(\underline{n}),$$

the set of subsets of the totally ordered set $\underline{n} = \{1, 2, , \dots, n\}$.

There is a unique poset isomorphism

$$\phi : \mathcal{P}(\underline{n}) \xrightarrow{\cong} \mathbf{1}^{\times n},$$

where $\mathbf{1}$ is the 2-element poset $0 \leq 1$. Here,

$$A \overset{\phi}{\mapsto} (\epsilon_1, \dots, \epsilon_n)$$

where $\epsilon_i = 1$ if and only if $i \in A$. We use the ordering of $\underline{n}$ to specify the poset isomorphism $\phi$.

# The box category

Suppose that $A \subset B \subset \underline{n}$. The **interval** $[A, B] \subset \mathcal{P}(\underline{n})$ is defined by

$$[A, B] = \{C \mid A \subset C \subset B\}.$$

There are canonical poset maps

$$\mathcal{P}(\underline{m}) \cong \mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(\underline{n}).$$

where $m = |B - A|$. These compositions are the coface maps $d : \square^m \subset \square^n$.

There are also co-degeneracy map $s : \square^n \to \square^r$, which are again determined by subsets $A \subset \underline{n}$, where $|A| = r$, and such that $s(B) = B \cap A$.

The cofaces and codegeneracies are the generators for the **box category** $\square$ consisting of the posets $\square^n$, $n \geq 0$, subject to the cosimplicial identities.

# Cubical sets

A **cubical set** is a functor $X : \Box^{op} \to \mathbf{Set}$.

$\Box^n \mapsto X_n$, and $X_n$ is the set of $n$-**cells** of $X$.

The collection of all such functors and natural transformations between them is the category $c\mathbf{Set}$ of cubical sets.

**Examples**

1) The **standard** $n$-**cell** $\Box^n$ is the functor $\hom(\ ,\Box^n)$ represented by $\Box^n = \mathcal{P}(\underline{n})$ on the box category $\Box$.

The $n$-cells of a cubical set $X$ can be identified with cubical set maps $\sigma : \Box^n \to X$.

2) Deleting the top cell from $\Box^n$ gives the **boundary** $\partial\Box^n$.

There are 2 maximal faces of $\partial\Box^n$ for each $i \in \underline{n}$: $[\{i\},\underline{n}]$, $[\emptyset,\{1,\dots,\hat{i},\dots,n\}]$.

3) The **cubical horn** $\sqcap^n_{(i,\epsilon)}$ is defined by deleting a single top face from $\partial\Box^n$.

# Higher dimensional automata

A **finite cubical complex** is a subcomplex $K \subset \square^n$. It is completely determined by cells

$$\square^r \subset K \subset \square^n$$

where the composites are cofaces. A cell is **maximal** if $r$ is maximal wrt these constraints.

Finite cubical complexes are **higher dimensional automata**.

# Triangulation

There is a **triangulation functor**

$$|\cdot| : c\mathbf{Set} \to s\mathbf{Set},$$

with $|\square^n| := B(\mathbf{1}^{\times n}) \cong (\Delta^1)^{\times n}$.

$B(C)$ is the **nerve** of a category $C$: $B(C)_n$ is the set

$$a_0 \to a_1 \to \cdots \to a_n$$

of strings of arrows of length $n$ in $C$.

The **triangulation** $|K|$ is defined by

$$|K| = \varinjlim_{\square^n \to K} |\square^n|,$$

indexed over the **cell category** $\square/K$ of $K$.

# Examples

1) $|\square^2| = B(\mathbf{1}^{\times 2}) = \Delta^1 \times \Delta^1$:

$$\begin{array}{ccc} (0,1) & \rightarrow & (1,1) \\ \uparrow & \nearrow\!\!\cdots & \uparrow \\ (0,0) & \rightarrow & (1,0) \end{array}$$

2) $|\square^1 \times \square^1|$ has 1-skeleton

$$\begin{array}{ccc} (0,1) & \rightarrow & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \rightarrow & (1,0) \end{array}$$

with non-degenerate 2-cells $\mathcal{P}(\underline{2}) \to \mathcal{P}(\underline{1}) \times \mathcal{P}(\underline{1})$ given by factorwise intersections $(\{1\},\{2\})$, $(\{2\},\{1\})$.

$$|\square^1 \times \square^1| \simeq S^2 \vee S^1$$

# Singular functor

The triangulation functor has a right adjoint,

$$S : s\mathbf{Set} \to c\mathbf{Set}$$

called the **singular** functor.

$S(X)$ has $n$-cells

$$S(X)_n = \hom((\Delta^1)^{\times n}, X).$$

**Example**: $S(BC)$ is the **cubical nerve** of a small category $C$.

# Standard homotopy theory

Say that a monomorphism of cubical sets is a **cofibration**.

A map $X \to Y$ of cubical sets is a **weak equivalence** if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of simplicial sets.

**Fibrations** of cubical sets are defined by a right lifting property with respect to all trivial cofibrations.

**Theorem 1.**

1) *With these definitions the category* $c\mathbf{Set}$ *has the structure of a proper, closed (cubical) model category.*
2) *The adjoint functors*

$$| \cdot | : c\mathbf{Set} \leftrightarrows s\mathbf{Set} : S$$

*define a Quillen equivalence.*

The triangulation functor does not preserve products. Here's the fix:

Define

$$\square^n \otimes \square^m := \square^{n+m}$$

Set

$$K \otimes \square^m := \varinjlim_{\square^n \to K} \square^n \otimes \square^m.$$

The cubical function complex $\mathbf{hom}(X, Y)$ has $n$-cells given by morphisms

$$X \otimes \square^n \to Y.$$

This construction satisfies the analog of Quillen's axiom **SM7**.

The nerve functor $B : \mathbf{cat} \to s\mathbf{Set}$ has a left adjoint

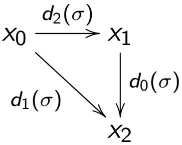
$$P : s\mathbf{Set} \to \mathbf{cat},$$

called the **path category** functor.

The path category $P(X)$ for $X$ is the category generated by the 1-skeleton $\mathrm{sk}_1(X)$ (a graph), subject to the relations:

1) $s_0(x)$ is the identity morphism for all vertices $x \in X_0$,

2) the triangle




commutes for all 2-simplices $\sigma : \Delta^2 \to X$ of $X$.

# Execution paths

Suppose that $K \subset \square^n$ is an HDA, with states (vertices) $x, y$. Then

$$P(|K|)(x,y)$$

is the set of **execution paths** from $x$ to $y$.

$P(K) := P(|K|)$ is the **path category** of the cubical complex $K$.

$P(K)$ can be defined directly for $K$: it is generated by the graph $\mathrm{sk}_1(K)$, subject to the relations given by $s_0(x) = 1_x$ for vertices $x$, and by forcing the commutativity of

$$\begin{array}{ccc} x_{\emptyset} & \longrightarrow & x_{\{1\}} \\ \downarrow & \searrow & \downarrow \\ x_{\{2\}} & \longrightarrow & x_{\{1,2\}} \end{array}$$

for each non-degenerate 2-cell $\sigma : \square^2 \subset K$ of $K$.

# Preliminary facts

**Lemma 2.**

$\mathrm{sk}_2(X) \subset X$ *induces* $P(\mathrm{sk}_2(X)) \cong P(X)$ *for simplicial sets (or cubical complexes)* $X$.

**Lemma 3.**

$\epsilon : P(BC) \to C$ *is an isomorphism for all small categories* $C$.

**Lemma 4.**

*There is an isomorphism* $G(P(X)) \cong \pi(X)$ *for all simplicial sets* $X$.

**Corollary 5 (Quillen).**

*There is an isomorphism* $\pi(BC) \cong G(C)$ *for all small categories* $C$.

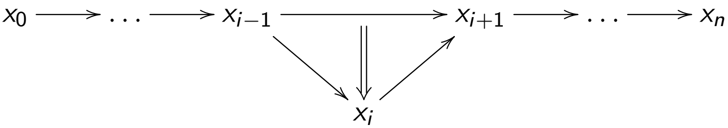
The Corollary was originally used to prove the isomorphism $\pi_1(BQ(\mathbf{M})) \cong K_0(\mathbf{M})$ for exact categories $\mathbf{M}$.

$L =$ finite simplicial complex: *$P(L)$ is the path component category of a 2-category $P_2(L)$.*

$P_2(L)$ consists of categories $P_2(L)(x, y)$, $x, y \in L$.
The objects (1-cells) are paths of non-deg. 1-simplices

$$x = x_0 \to x_1 \to \cdots \to x_n = y$$

of $L$. The morphisms of $P_2(L)(x, y)$ are composites of the pictures




where the displayed triangle bounds a non-deg. 2-simplex.
Compositions are functors

$$P_2(L)(x, y) \times P_2(L)(y, z) \to P_2(L)(x, z)$$

defined by concatenation of paths.

**Theorem 6.**

*Suppose that L is a finite simplicial complex. Then $P_2(L)$ is a "resolution" of the path category $P(L)$ in the sense that there is an isomorphism*

$$\pi_0 P_2(L) \cong P(L).$$

$\pi_0 P_2(L)$ is the **path component category** of $P_2(L)$. Its objects are the vertices of $L$, and

$$\pi_0 P_2(L)(x,y) = \pi_0(BP_2(L)(x,y)).$$

# The algorithm

Here's an algorithm for computing $P(L)$ for $L \subset \Delta^N$, in outline:

1) Find the 2-skeleton $\mathrm{sk}_2(L)$ of $L$ (vertices, 1-simplices, 2-simplices).
2) Find all paths (strings of 1-simplices)

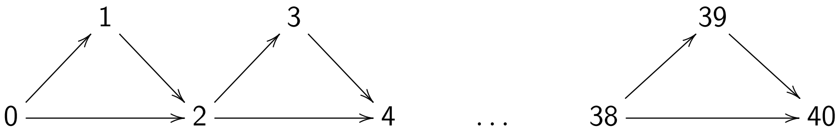
$$\omega : v_0 \xrightarrow{\sigma_1} v_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_k} v_k$$

in $L$.
3) Find all morphisms in the category $P_2(L)(v, w)$ for all vertices $v < w$ in $L$ (ordering in $\Delta^N$).
4) Find the path components of all $P_2(L)(v, w)$, by approximating path components by full connected subcategories, starting with a fixed path $\omega$.

Code: Graham Denham (Macaulay 2), Mike Misamore (C).

# Example: the necklace

Let $L \subset \Delta^{40}$ be the subcomplex




This is 20 copies of the complex $\partial\Delta^2$ glued together. There there are $2^{20}$ morphisms in $P(L)(0, 40)$.

**Moral**: The size of the path category $P(L)$ can grow exponentially with $L$.

The code for this example runs on a desktop with at least 5 GB of memory. The listing of paths consumes 2 GB of disk.

There is nothing that you can do to make it simpler — there are no 2-simplices.

# Complexity reduction

Suppose that $L \subset K \subset \Delta^N$.

$L$ is a **full subcomplex** of $K$ if the following hold:

1) $L$ is path-closed in $K$, in the sense that, if there is a path

$$v = v_0 \to v_1 \to \cdots \to v_n = v'$$

in $K$ between vertices $v, v'$ of $L$, then all $v_i \in L$,

2) if all the vertices of a simplex $\sigma \in K$ are in $L$ then the simplex $\sigma$ is in $L$.

**Lemma 7.**

*Suppose that $L$ is a full subcomplex of $K$. Then the functor $P(L) \to P(K)$ is fully faithful.*

ie. $x, y \in L$: $P(L)(x, y) = P(K)(x, y)$.

# Examples

- $\partial\Delta^2 \overset{d^0}{\subset} \Lambda^3_0$ and $\partial\Delta^2 \overset{d^3}{\subset} \Lambda^3_3$ are full subcomplexes.
- Suppose that $i \leq j$ in $\mathbf{N}$. $K[i,j]$ is the subcomplex of $K$ such that $\sigma \in K[i,j]$ if and only if all vertices of $\sigma$ are in the interval $[i,j]$ of vertices $v$ such that $i \leq v \leq j$. $K[i,j]$ is a full subcomplex of $K$.
- Suppose that $v \leq w$ are vertices of $K$. Let $K(v,w)$ be the subcomplex of $K$ consisting of simplices whose vertices appear on a path from $v$ to $w$. $K(v,w)$ is a full subcomplex of $K$.

Construct $K(v,w)$ from $K[v,w]$ by deleting *sources* and *sinks*.

A vertex $v$ is a **source** of $K$ if there are no 1-simplices $u \to v$ in $K$. (0 is a source of $\Lambda^3_0$)

A vertex $z$ is a **sink** if there are no 1-simplices $z \to w$ in $K$. (3 is a sink of $\Lambda^3_3$)

# Corners

Suppose that $K \subset \square^n$ is a cubical complex. Say that a vertex $x$ is a **corner** of $K$ if it belongs to only one maximal cell.

**Lemma 8 (Misamore).**

*Suppose that $x$ is a corner of $K$, and let $K_x$ be the subcomplex of cells which do not have $x$ as a vertex. Then the induced functor*

$$P(K_x) \to P(K)$$

*is fully faithful.*

There are two steps in the proof [4]:

- Suppose that $x$ is a vertex of the cell $\square^r$ and let $\square^r_x \subset \square^r$ be the subcomplex of cells which do not have $x$ as a vertex. Then $P(\square^r_x) \to P(\square^n)$ is fully faithful.

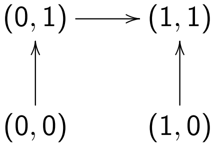
- Suppose that $x$ is a corner of $K$, and that $x$ is a vertex of a maximal cell $\square^r \subset K$. Let $K_x \subset K$ be the subcomplex whose cells do not have $x$ as a vertex. Then the diagram

$$\begin{array}{ccc} P(\square^r_x) & \longrightarrow & P(K_x) \\ \downarrow & & \downarrow \\ P(\square^r) & \longrightarrow & P(K) \end{array}$$

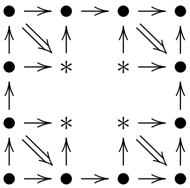
  is a pushout, so that $P(K_x) \to P(K)$ is fully faithful.

This uses an assertion of Fritsch and Latch that fully faithful functors are closed under pushout.

# Examples

1) The cubical horn  has a sink but no corners.



2) The Swiss flag  has 6 corners, 1 sink, 1 source.

# Going beyond

1) These algorithms depend on having an entire HDA in storage, in a computer system that is powerful enough to analyze it.

2) We want local to global methods to study large (aka. "infinite") models with patching techniques. One could parallelize existing algorithms by computations on patches.

3) Nobody has any idea of what higher homotopy invariants should mean for higher dimensional automata, or even what the appropriate homotopy theory should be.

Suggestion: Joyal's homotopy theory of quasi-categories.

$K \mapsto P(K)$ is a quasi-category homotopy invariant, but beware: if $K \to L$ is a quasi-category weak equivalence then $P(K) \to P(L)$ is an **equivalence** of categories.

A **quasi-category** is a simplicial set $X$ which satisfies the extension condition

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

for the **inner horns** $\Lambda^n_k$, $0 < k < n$.

**Example**: If $C$ is a small category, then $BC$ is a quasi-category. All standard simplices $\Delta^n = B(\underline{n})$ are quasi-categories.

The extension problem

$$\begin{array}{ccc} \Lambda^2_1 & \xrightarrow{(\beta,\ ,\alpha)} & BC \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array}$$

is solved by the 2-simplex $\sigma = (\alpha, \beta)$ of $BC$. $d_1(\sigma) = \beta \cdot \alpha$.

# Intervals and homotopies

Set $I = B\pi(\Delta^1)$. $\pi(\Delta^1)$ is the trivial groupoid on objects $0, 1$.

An $I$-**homotopy** between maps $f, g : X \to Y$ is a diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle in_0}\downarrow & \searrow^{f} & \\ X \times I & \longrightarrow & Y \\ {\scriptstyle in_1}\uparrow & \nearrow_{g} & \\ X & & \end{array}$$

The set of all $I$-homotopies between maps $X \to Y$ generates the $I$-homotopy relation.

$\pi_I(X, Y)$ is the set of $I$-homotopy classes.

# Quasi-category weak equivalences

A map $f : X \to Y$ is a **quasi-category weak equivalence** if the induced map

$$f^* : \pi_I(Y, Z) \to \pi_I(X, Z)$$

is a bijection for all quasi-categories $Z$. (Joyal: "categorical weak equivalence")

1) A map $f : X \to Y$ of quasi-categories is a quasi-category weak equivalence if and only if it is an $I$-homotopy equivalence.

2) All pushouts of inner horn inclusions $\Lambda^n_k \subset \Delta^n$ are quasi-category weak equivalences.

3) There is a natural map $j : X \to LX$ where $LX$ is a quasi-category and $j$ is a sequence of pushouts of inner horn inclusions. This map $j$ is a quasi-category weak equivalence.

4) $f : X \to Y$ is a quasi-category weak equivalence if and only if the induced map $LX \to LY$ is an $I$-homotopy equivalence.

Say that a monomorphism of simplicial sets is a **cofibration**, as usual.

**Theorem 9 (Joyal, Cisinski).**

*The cofibrations and quasi-category weak equivalences give s***Set** *the structure of a left proper, cofibrantly generated, closed cubical model category.*

**Fact**: The inner horn inclusions $\Lambda^n_k \to \Delta^n$ induce isomorphisms $P(\Lambda^n_k) \xrightarrow{\cong} P(\Delta^n)$ of path categories.

The map $j : X \to LX$ induces an isomorphism $P(X) \xrightarrow{\cong} P(LX)$.

The path category functor $P$ preserves finite products, and $P(I) = \pi(\Delta^1)$.

**Fact**: If $f : X \to Y$ is a quasi-category weak equivalence then $f_* : P(X) \to P(Y)$ is an equivalence of categories.

# Local to global constructions

$op|_T$ is the site of open subsets of a topological space $T$.
$s\,\mathrm{Pre} := s\,\mathrm{Pre}(op|_T)$ is the category of simplicial presheaves on $op|_T$.

Say that a monomorphism of simplicial presheaves is a **cofibration**.

Say that a morphism $X \to Y$ of simplicial presheaves is a **local quasi-category weak equivalence** if it induces quasi-category weak equivalences $X_x \to Y_x$ in all stalks.

**Theorem 10 (Meadows).**

*The cofibrations and quasi-category weak equivalences give* $s\,\mathrm{Pre}(op|_T)$ *the structure of a left proper, cofibrantly generated, closed cubical model category.*

We're convinced that there is a generalization of this result to simplicial presheaves on an arbitrary Grothendieck site.

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