Curvature-free estimates for solutions of variational problems in Riemannian geometry

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$$l \leq C(n)\ diameter(M)?$$

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Question: Can we majorize their lengths in terms of the diameter of M?

Theorem

(F. Almgren-J. Pitts) Let M be a closed Riemannian manifold of dimension $n \in \{3, 4, 5, 6, 7\}$. Then there exists an embedded smooth minimal hypersurface in M.

This result can be generalized to other dimensions and codimensions if one does not insist on the smoothness of the minimal object anymore.

1. Some quantitative versions of Fet-Lyusternik theorem.

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l denotes the length of a shortest non-constant periodic geodesic.
An obvious observation: If M is nonsimply-connected, then $l \leq 2d$, (d denotes diameter of M) (Exercise).

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Problem. GUESS a Riemannian metric on S^2 for which $\frac{l}{d}$ is (nearly) maximal possible.

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Conjectured optimal shape (E.Calabi): Two equilateral triangles glued along their common boundary.

l and the volume of M: nonsimply-connected case.

Systolic geometry: Find an upper bound for the length of the shortest non-contractible periodic geodesic on M in terms of $vol(M)$.

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Theorem

(M. Gromov) If M^n is essential, then there exists a non-contractible periodic geodesic of length $\leq c(n)vol(M^n)^{\frac{1}{n}}$.

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(M. Gromov) If M^n is essential, then there exists a non-contractible periodic geodesic of length $\leq c(n)vol(M^n)^{\frac{1}{n}}$.

But, I. Babenko proved that this result holds only for essential manifolds.

Geodesic nets: Let M be a Riemannian manifold. A geodesic net in M is an immersed (multi)graph such that:

1) The image of each edge is a geodesic;

2) For each vertex v the sum of unit tangent vectors at v to all edges adjacent to v is equal to 0.

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Theorem

(A.N., R. Rotman) There exists (explicit) constants $c_1(n)$, $c_2(n)$ such that for each closed Riemannian manifold M^n the length of the shortest geodesic net on M does not exceed $c_1(n)d$. Also, it does not exceed $c_2(n)vol(M^n)^{\frac{1}{n}}$.

Theorem

(R. Rotman) The estimates in the previous theorem hold for the length of a shortest geodesic net that consists of at most $N(n)$ geodesic loops based at the same point.

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2. Is it true that for each closed Riemannian manifold M there exists a geodesic net on M which is not composed of periodic geodesics?

2. Quantitative Lyusternik-Shnirelman theorem.

Theorem

(Y. Liokumovich, A. N., R. Rotman)
Let M be a Riemannian 2-sphere. Then there exist three simple periodic geodesics on M such that their lengths do not exceed, correspondingly, $5d$, $10d$ and $20d$, where d denotes the diameter of M.

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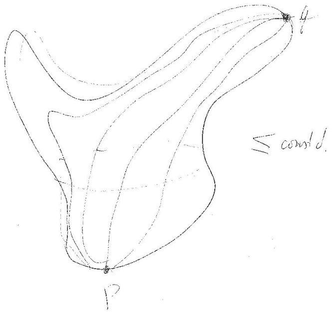
A very general idea of the proof: The original proof by Lyusternik and Shnirelman uses three specific cycles in the space of nonparametrized simple closed curves on M.
If M has a "nice" metric, then one can find homologous cycles that consist of "short" curves, and then the desired estimates follow from the existence proof.
If M is not "nice", its "ruggedness" implies the existence of "short" simple closed geodesics that are local minima of the length functional.

Here “nice” means that M can be sliced into pairwise non-intersecting nonself-intersecting curves of length $\leq const\ d$ connecting a pair of points. It turns out that one can use these curves to bound lengths of simple closed curves in some cycles representing each of the three homology classes in Lyusternik-Shnirelman proof.


q
≤ const.
p

Now one attempts to construct a slicing of M into nonself-intersecting curves of length $\leq$ *const d* that connect a fixed pair of points. Our construction process can be blocked only by a simple periodic geodesic of index 0 and "small" length. Each time the extension process is blocked, we can continue in a different fashion until it is blocked again. We are done after the appearance of three "obstructing" simple periodic geodesics.

Note that, in general, one cannot slice a Riemannian 2-sphere into closed curves of length $\leq$ *const* d. So, not all 2-spheres are "nice". For example:

Theorem

(Y. Liokumovich) There is no contant C such that each Riemannian 2-sphere of diameter d can be divided into two parts of equal area by a (not necessarily connected) closed curve of length $\leq Cd$.

So, the simple geodesics that we obtain are not necessarily those that appear in the original LS proof. If one wishes to majorize the lengths of three simple geodesics provided by the original proof, one can use the following theorem:

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Theorem

(Y. Liokumovich, A.N., R. Rotman) *Let M be a Riemannian 2-sphere of diameter d and area A. Then it can be sliced into simple loops of length* $\leq 200d \max\{1, \ln \frac{\sqrt{A}}{d}\}$. *The simple loops intersect only at their common base point. This upper bound is optimal up to a constant factor.*

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This theorem implies that three "original" LS simple periodic geodesics have length $\leq 800d \max\{1, \ln \frac{\sqrt{A}}{d}\}$.

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This theorem implies that three "original" LS simple periodic geodesics have length $\leq 800d\max\{1, \ln\frac{\sqrt{A}}{d}\}$.
This theorem answers a question of S. Frankel and M. Katz which was a modification of an earlier question posed by M. Gromov.

The strategy of the proof is to use cuts of several different types to reduce the problem to a similar "controlled" slicing problem for smaller and smaller subdiscs. The cuts come from the coarea formula, Besicovitch inequality and a version of Gromov's "attempted impossible extension" technique.

3. Quantitative versions of Serre's theorem

Theorem

(R. Rotman) Let M^n be a closed Riemannian manifold. For each $p \in M^n$ there exists a geodesic loop based at p of length $\leq 2nd$ (and even $\leq 2qd$, where $q = \min\{i | \pi_i(M^n) \neq 0\}$).

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Theorem

(A.N., R. Rotman) Let p, q be any two points on a closed Riemannian manifold M^n. For every m there exists m distinct geodesics connecting p and q of length $\leq 4m^2 nd$.

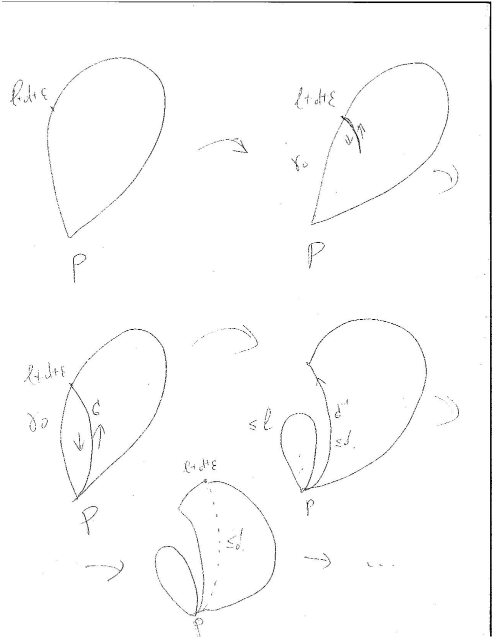
PROOF:

Curve-shortening process: It can be blocked only by many "short" geodesic loops.

Purpose: Given a curve γ connecting two points p and q we would like to shorten it by a path homotopy (=a homotopy that keeps p and q fixed).

Assumption: There are no geodesic loops based at p of length in the interval $(l, l + 2d]$ for some l.

Conclusion: There is a path homotopy that shortens γ to the length $\leq l + d$.



l+d+ε
P
l+d+ε
γ₀
P
l+d+ε
γ₀
σ
P
≤ l
σ⁻¹
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Process: First, note that there exists $\epsilon > 0$ such that there are no geodesics in the interval $(l, l + 2d + \epsilon]$ (by an easy compactness argument).

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Connect its endpoint with p by a minimizing geodesic σ (of length $\leq d$). Insert σ traversed twice in the opposite directions inside γ.

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Connect its endpoint with p by a minimizing geodesic σ (of length $\leq d$). Insert σ traversed twice in the opposite directions inside γ.
Shorten the loop $\gamma_0 * \sigma$ to a geodesic loop τ based at p by a path homotopy. The length of $\tau \leq l$. Curve γ shortens to $\tau * \sigma^{-1}*$the rest of γ that has length $\leq$ length$(\gamma) - \epsilon$.

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Repeat the process.
As l is arbitrary, one needs geodesic loops with length in intervals $(0, 2d]$, $(2d, 4d], \ldots$ to block the curve shortening process.

The process is not continuous, but one can still construct a parametric version.

Theorem

(A.N., R. Rotman) Let M^n be a closed Riemannian manifold, $p \in M^n$. Then either
1) there exist k geodesic loops of index 0 based at p with lengths in the intervals $(0, 2d],\ (2d, 4d], \ldots, (2(k-1)d, 2kd]$,
or
2) For each N any map of S^N into the space of based loops $\Omega_p M^n$ can be homotoped to its subspace $\Omega_p^L M^n$ that consists of loops of length $\leq L = 4(k+2)(N+1)d$.

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The proof of Serre's theorem given by Albert Schwartz imples that the length of kth geodesic between p and q does not exceed $c(M^n)k$, where $c(M^n)$ depends on the Riemannian metric on M^n in an unknown way.
Problem: Is it true that the length of the kth geodesic does not exceed $c(n)kd$, where $c(n)$ depends only on n?

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Theorem

(A.N., R. Rotman) If $n = 2$ then the length of the kth geodesic between p and q does not exceed $22kd$.

Problem. Is there an upper bound for the length of the first k geodesics between p and q of the form $c(k)d$ (that is, there is no dependence on n)?.

4. Quantiative versions of Almgren-Pitts theorem.

Definition: Let M be a Riemannian manifold such that $H_1(M)$ is trivial. For each $x > 0$ the first homological filling function of M is defined as the infimum of y such that each closed curve of length $\leq x$ can be represented as the boundary of a singular Lipschitz chain $c = \Sigma_i a_i \sigma_i$ such that the area$(c) = \Sigma_i |a_i|$area$(\sigma_i) \leq y$.

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Theorem

(A.N., R. Rotman) Let M be a Riemannian homology 3-sphere (e.g. S^3). The smallest area of an embedded minimal surface in M does not exceed (i) $6F_1(2d)$; (ii) $12F_1(3300\ vol(M)^{\frac{1}{3}})$.

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Problem. Is it true that each closed Riemannian 3-dimensional manifold of volume 1 contains a smooth embedded minimal surface of area $\leq 10^{10}$?

Theorem

(P. Glynn-Adey, Y. Liokumovich): A closed Riemannian manifold M^n of dimension $n \in \{3,4,5,6,7\}$ satisfying $Ric \geq -(n-1)a^2$ for $a \geq 0$ contains a closed smooth embedded minimal hypersurface Σ of volume $\leq C(n)\max\{1, a\ vol(M^n)^{\frac{1}{n}}\} vol(M^n)^{\frac{n-1}{n}}$.

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Their upper bound is for the $(n-1)$-width of M and holds for all n. It is a corollary of a stronger upper bound for the $(n-1)$-width that involves $n, vol(M^n)$ and the "minimal conformal volume" of M^n, which is a scale-invariant conformal invariant.

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Note that there is no upper bound on the $(n-1)$-width of M in terms of $vol(M)$, if $n > 2$ (Larry Guth; D. Burago and S. Ivanov).

Theorem

(P. Glynn-Adey, Y. Liokumovich): A closed Riemannian manifold M^n of dimension $n \in \{3,4,5,6,7\}$ satisfying $Ric \geq -(n-1)a^2$ for $a \geq 0$ contains a closed smooth embedded minimal hypersurface Σ of volume $\leq C(n)\max\{1, a\ vol(M^n)^{\frac{1}{n}}\}vol(M^n)^{\frac{n-1}{n}}$.

Their upper bound is for the $(n-1)$-width of M and holds for all n. It is a corollary of a stronger upper bound for the $(n-1)$-width that involves $n, vol(M^n)$ and the "minimal conformal volume" of M^n, which is a scale-invariant conformal invariant.
Note that there is no upper bound on the $(n-1)$-width of M in terms of $vol(M)$, if $n > 2$ (Larry Guth; D. Burago and S. Ivanov). So, one cannot hope for curvature-free estimates for $(n-1)$-widths.

Problem. Are there analogous results for higher codimensions?

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(P. Glynn-Adey, Y. Liokumovich); an effectice version of the existence theorem by Fernando Coda Marquees and André Neves

Let M^n be a closed Riemannian manifold of dimension $n \in \{3, 4, 5, 6, 7\}$ with positive Ricci curvature. Then for each $k = 1, 2, \ldots$ it contains at least k distinct minimal hypersurfaces of volume $\leq C(n) \frac{vol(M^n)}{minvol_{n-1}(M^n)^{\frac{1}{n-1}}} k^{\frac{1}{n-1}}$, where $minvol_{n-1}(M^n)$ denotes ther minimal volume of a non-trivial minimal hypersurface in M^n.

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Question. Is it possible to get rid of $minvol_{n-1}(M^n)$ in this estimate?

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2. Assume that this class is represented by a map f of, say, a sphere S^m to M. We can attempt an (impossible) extension of f to a disc D^{m+1} triaingulated as a cone over S^m. Induction is done by induction with respect to skeleta.

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3. Each step is an extension in M, but we try to represent it as an extension in $X(M)$ so that the image of the extension consists "small" objects in $X(M)$. If an extension in $X(M)$ is impossible, then there is a "small" extremal object in $X(M)$ obstructing the extension process.

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4. If the extension process is unobstructed up to the dimension m , then the boundary of at least one of $(m+1)$-cells of D^{m+1} represents a non-trivial cycle. Its boundary had been mapped into "small" objects in $X(M)$, and its contractibility is obstructed by a "small" minimal object in $X(M)$.

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5. Another useful "attempted impossible extension" (Gromov): Embed $M = M^n$ into $L^\infty(M)$ using Kuratowski embedding. Represent M^n as ∂W^{n+1}, where W is an $c(n)vol^{\frac{1}{n}}(M^n)$-neighborhood of M^n (Gromov's filling radius theorem). Triangulate W^{n+1} into small simplices, and attempt to extend the identity map $M^n \longrightarrow M^n$ to a map of W^{n+1} into M^n. First, one sends all vertices to closest points of M^n, then 1-simplices to minimal geodesics, setting the scale for subsequent steps of the extension process as $const(n)vol(M^n)^{\frac{1}{n}}$.

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6. Assume that one needs to establish upper bounds not just for one minimal object in M but for a finite or infinite family of minimal objects. If one has a sweep-out of a class of M by “small” loops or cycles, one typically gets the desired estimate not for just one minimal object but for all of them. On the other hand, the extension process can be obstructed by just one “small” minimal object. The idea is to start the extension process anew looking for maps into “bigger” objects in $X(M)$. The idea is that either we are going to get “bigger” and “bigger” obstructing minimal objects, or we will get a desired “controlled” sweep-out.