# Uniqueness and purity in multi-agent matching problems

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# Multi-agent matching under transferable utility

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- Matching measure:
  - A probability measure $\gamma$ on $X_1 \times X_2 \times ... \times X_m$ whose marginals are the $\mu_i$.
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  - $\Gamma(\mu_1, \mu_2, ..., \mu_m)$ = set of all matchings.
- A matching is stable if there exists functions $u_1(x_1), u_2(x_2), ..., u_m(x_m)$ such that

$$\sum_{i=1}^{m} u_i(x_i) \geq s(x_1, x_2, ..., x_m)$$

with equality $\gamma$ almost everywhere (payoff functions).

  - A division of the utility among matched agents.

# Variational formulation: multi-marginal optimal transport

- Shapley-Shubik (1972): A matching is stable if and only if it maximizes:

$$\gamma \mapsto \int_{X_1 \times X_2 \times \ldots \times X_m} s(x_1, x_2, ..., x_m) d\gamma,$$

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- When $m = 2$, the generalized Spence-Mirrlees, or twist condition yields uniqueness and purity:
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- Brenier '87, Gangbo '95, Caffarelli '96, Gangbo-McCann '96, Levin '96: If $\mu_1$ is absolutely continuous with respect to Lebesgue measure and $s$ is twisted, the stable match $\gamma$ is unique and pure.

# A condition for purity and uniqueness

- A set $S \subseteq X_2 \times X_3 ... \times X_m$ is an $s$-splitting set at a fixed $x_1 \in X_1$ if there exist functions $u_2(x_2), ..., u_m(x_m)$ such that $\sum_{i=2}^{m} u_i(x_i) \geq s(x_1, ..., x_m)$ with equality on $S$.

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- We say $s$ is *twisted on splitting sets* if whenever $S \subseteq X_2 \times X_3 ... \times X_m$ is a splitting set at $x_1$,

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- Kim-P (2013) : If $\mu_1$ is absolutely continuous with respect to Lebesgue measure and $s$ is twisted on splitting sets, the stable match $\gamma$ is unique and pure.

# Example: One dimensional case

- $m = 2$: Recall classical (two marginal, one dimension) Spence-Mirrlees condition (supermodularity): $\frac{\partial^2 s}{\partial x_1 \partial x_2} > 0$ – leads to positive assortative matching.
  - $\frac{\partial^2 s}{\partial x_1 \partial x_2} < 0$ (submodularity) is also twisted – leads to negative assortative matching.

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- When $x_1, x_2, ..., x_m \in \mathbb{R}$, twist on splitting sets is essentially equivalent to:

$$\frac{\partial^2 s}{\partial x_i \partial x_j}\left[\frac{\partial^2 s}{\partial x_k \partial x_j}\right]^{-1}\frac{\partial^2 s}{\partial x_k \partial x_i} > 0$$

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- *Satisfied* for supermodular costs: $\frac{\partial^2 s}{\partial x_i \partial x_j} > 0$ for all $i \neq j$. These surpluses were studied by Carlier (2003) – lead to positive assortative matching.
- *Violated* for submodular costs: $\frac{\partial^2 s}{\partial x_i \partial x_j} < 0$ for all $i \neq j$.

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- Under mild conditions, $s$ satisfies twist on splitting sets.
    - Ex. $s(x_1, x_2, ..., x_m) = \sum_{i,j=1}^m x_j \cdot x_i$ (Gangbo and Swiech (1998)).

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- When all the $\mu_i$ are the same, the only pure, symmetric matching is concentrated on the diagonal.
- For $s(x_1, ...x_m) = -\sum_{i \neq j}^{m} x_i \cdot x_j$, measures supported on the surface $\{\sum_{i=1}^{m} x_i = 0\}$ are optimal for their marginals.

# References

- A general condition for Monge solutions in the multi-marginal optimal transport problem, with Young-Heon Kim. *SIAM J. Math. Anal.* 46 (2014) 1538-1550.
- Multi-marginal optimal transport: theory and applications. To appear in *ESAIM: Math. Model. Numer. Anal.* (Special issue on "Optimal transport in applied mathematics.")

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www.ualberta.ca/$\sim$pass/papers/