Mean Field Games and Stochastic Growth Modeling

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Background

- Mean field games: Competitive decision with a large no. of agents
  - "An interacting N-particle system". Then let $N \to \infty$
    - Caines, Huang, and Malhamé (03, 06, ...); Lasry and Lions (06, 07, ...); an overview by Bensoussan et. al. (2012); Buckdahn et. al. (2011); a survey by Gomes and Saúde (2013)
  - Early ideas in economic literature: Jovanovic and Rosenthal (Anonymous sequential games, 1988); continuum population modeling, finite MDP
- Stochastic growth theory
  - Optimal control of a whole sector of an economy
    - The pioneering work (Brock and Mirman, J. Econ. Theory, 1972); a nice survey (Olson and Roy, 2006)
    - Continuous time (Merton, 1975)
  - More generally: Nash games of N producers (e.g., Amir, Games Econ. Behav., 1996). Example: several firms in the fishery industry

The start of growth theory: deterministic root

Frank Ramsey (1903-1930)

- F. P. Ramsey. A mathematical theory of saving. *The Economic Journal*, vol. 38, no. 152, pp. 543-559, 1928.

Early motivation in engineering

- N wireless users; x_i: channel gain (in dB); p_i: power. Continuous time channel modeling: Charalambous et al (1999)
- objective for SIR (signal-to-interference ratio):

$$\frac{e^{x_i} p_i}{\frac{\alpha}{N}\sum_{j\neq i} e^{x_j} p_j + \sigma^2} \approx \gamma_{\rm target}$$

σ^2: thermal noise; $\frac{1}{N}$ is due to using a spreading gain whose length is proportional to the user number

- Dynamic game

$$dx_i = a(\mu - x_i)dt + C dW_i$$

$$dp_i = u_i dt$$

$$J_i = E\int_0^T \left\{ \left[e^{x_i} p_i - \gamma_{\rm target}\left(\frac{\alpha}{N}\sum_{j\neq i} e^{x_j} p_j + \sigma^2\right)\right]^2 + r u_i^2 \right\} dt$$

Early motivation from engineering

Nonlinear dynamic game

$$dx_i = a(\mu - x_i)dt + CdW_i$$
$$dp_i = u_i dt$$
$$J_i = E\int_0^T \left\{ \left[e^{x_i} p_i - \gamma_{\rm target}\left(\frac{\alpha}{N} \sum_{k\neq i} e^{x_k} p_k + \sigma^2 \right) \right]^2 + r u_i^2 \right\} dt$$

$\Longrightarrow$

Linear-Quadratic-Gaussian mean field game theory

$$dx_i = (a_i x_i + b u_i)dt + CdW_i$$
$$J_i = E\int_0^T \left\{ \left[x_i - \gamma\left(\frac{1}{N} \sum_{j\neq i} x_j + \eta \right) \right]^2 + r u_i^2 \right\} dt$$

Even such a simple model is interesting enough! (HCM'03, 04, 07)

Early motivation from engineering

Linear-Quadratic-Gaussian mean field game theory

$$dx_i = (a_i x_i + bu_i)dt + CdW_i$$

$$J_i = E\int_0^T \{[x_i - \gamma(\frac{1}{N}\sum_{j\neq i} x_j + \eta)]^2 + ru_i^2\}dt$$

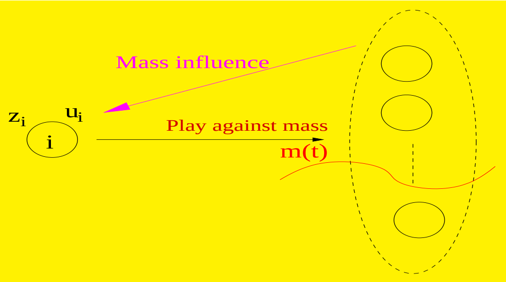
Fundamental issues:

- Existing theory yields Nash strategies of the form $u_i(t, x_1, \ldots, x_N)$
- Informational requirement is too high!
- Hope to design strategies of the form

  $u_i(t,$ "local state" x_i, "macoroscopic effect")

- How well such decentralized strategies perform in the original N player game?

Mean field game: one against the MASS



- Everyone plays against m_t (freeze it!), giving optimal responses
  - m_t can appear as a measure, first order statistic (mean), etc.
- The optimal responses regenerate m_t when no. of players $N \to \infty$

The basic framework of MFGs

$$\begin{cases} P_0\text{—Game with } N \text{ players; Example} \\ dx_i = f(x_i, u_i, \delta_x^{(N)})dt + \sigma(\cdots)dw_i \\ J_i(u_i, u_{-i}) = E\int_0^T l(x_i, u_i, \delta_x^{(N)})dt \\ \delta_x^{(N)} : \text{empirical distribution of } (x_j)_{j=1}^N \end{cases} \quad \xrightarrow{\text{solution}} \quad \begin{array}{l} \text{Coupled Hamilton-Jacobi-Bellman system} \\ u_i = u_i(t, x_1, \ldots, x_N), 1 \le i \le N \\ \text{Centralized strategy!} \end{array}$$

$\downarrow$*construct* $\nwarrow$*performance?* $\downarrow N \to \infty$

$$\begin{cases} P_\infty\text{—Limiting problem, 1 player} \\ dx_i = f(x_i, u_i, \mu_t)dt + \sigma(\cdots)dw_i \\ \bar{J}_i(u_i) = E\int_0^T l(x_i, u_i, \mu_t)dt \\ \text{Freeze } \mu_t, \text{ as approx. of } \delta_x^{(N)} \end{cases} \quad \xrightarrow{\text{solution}} \quad \begin{cases} \hat{u}_i(t, x_i) : \text{ optimal response} \\ \text{HJB } (v(T, \cdot) \text{ given}) : \\ -v_t = \inf_{u_i}(f^T v_{x_i} + l + \frac{1}{2}Tr[\sigma\sigma^T v_{x_i x_i}]) \\ \text{Fokker-Planck-Kolmogorov} : \\ p_t = -div(fp) + \sum((\frac{\sigma\sigma^T}{2})_{jk} p)_{x_i^j x_i^k} \\ \text{Coupled via } \mu_t \text{ (w. density } p_t, p_0 \text{ given)} \end{cases}$$

- The consistency based approach (red) is more popular; related to ideas in statistical physics (McKean-Vlasov equation), FPK may appear as MV-SDE
- When a major player or common noise appears, new tools (stochastic mean field dynamics, master equation, etc) are needed

Further major issues

- Major-minor players instead of peers in the mean field game
  - Motivation: institutional traders, large corporations, power generators (with respect to residential consumers), etc
- Mean field teams (cooperative social optimization) instead of games
- Robustness with model uncertainty
- ...

Application of MFGs to economic growth, finance, ...

- Guéant, Lasry and Lions (2011): human capital optimization
- Lucas and Moll (2011): Knowledge growth and allocation of time (JPE in press)
- Carmona and Lacker (2013): Investment of n brokers
- Espinosa and Touzi (2013): Optimal investment with relative performance concern (depending on $\frac{1}{N-1}\sum_{j\neq} X_j$)
- Chan and Sircar (2014): Bertrand and Cournot MFGs (coupling via average prices or quantities)
- Jaimungal (2014): Optimal execution with major-minor agents in trading (liquidation).
- ......

Organization of the talk

- Discrete time
  - We extend the neo-classical growth model (pioneered by Brock and Mirman 1972; see a comprehensive survey by Olson and Roy, 2006) to the mean field setting
- Continuous time
  - The classical SDE modeling by Merton (1975)
  - Stochastic depreciation: Walde (J. Econ. Dyn. Control, 2011); Feicht and Stummer (2010)
  - Our mean field modeling is based on the above works (Huang and Nguyen, to be presented at IEEE CDC'14)

Classical stochastic growth model: Review

The one-sector economy at stage t involves two basic quantities:

- κ_t: the capital stock (used for production); c_t: consumption

The next stage output y_{t+1}:

$$y_{t+1} = f(\kappa_t, r_t), \quad t = 0, 1, \dots,$$

- $f(\cdot,\cdot)$: production function; r_t: random disturbance; y_0: given
- $\kappa_t + c_t = y_t$

Objective: maximize the utility functional $E\sum_{t=0}^{\infty} \rho^t \nu(c_t)$;
$\nu(c_t)$: utility from consumption, usually concave on $[0, \infty)$

Brock and Mirman (J. Econ. Theory, 1972) pioneered stochastic growth theory.

Notation in the mean field model

Keep track of the notation (for the main part):

u_t^i:	control (allocation for capital stock)
X_t^i:	state (production output)
N:	number of players in the game
c_t^i:	consumption
$V_i(x,t)$:	value function
$G(p,W), g$:	growth coefficient in production
W:	white noise
p:	aggregate capital stock
γ:	HARA utility exponent

Mean field production dynamics of N agents

- X_t^i: output (or wealth) of agent i, $1 \le i \le N$
- $u_t^i \in [0, X_t^i]$: capital stock
- $c_t^i = X_t^i - u_t^i$: consumption; W_t^i: random disturbance
- $u_t^{(N)} = (1/N)\sum_{j=1}^N u_t^j$: aggregate capital stock

The next stage output, measured by the unit of capital, is

$$X_{t+1}^i = G(u_t^{(N)}, W_t^i)u_t^i, \qquad t \ge 0, \tag{3.1}$$

Motivation for the mean field production dynamics:

- Use $u_t^{(N)}$ as a proxy of the macroscopic behavior of the population.
- Congestion effect – Barro and Sala-I-Martin (Rev. Econ. Stud., 1992); Liu and Turnovsky (J. Pub. Econ., 2005). They consider static models of a finite number of firms.

The utility functional

The utility functional is

$$J_i(u^i, u^{-i}) = E\sum_{t=0}^{T}\rho^t v(X_t^i - u_t^i),$$

- $\rho \in (0,1]$: the discount factor
- $c_t^i = X_t^i - u_t^i$: consumption, $\quad u^{-i} = (\cdots, u^{i-1}, u^{i+1}, \cdots)$

We take the HARA utility

$$v(z) = \frac{1}{\gamma}z^\gamma, \quad z \geq 0, \qquad \gamma \in (0,1).$$

Assumptions

(A1) (i) Each sequence $\{W_t^i, t \in \mathbb{Z}_+\}$ consists of i.i.d. random variables with support D_W and distribution function F_W. The N sequences $\{W_t^i, t \in \mathbb{Z}_+\}$, $i = 1, \ldots, N$ are i.i.d. (ii) $\{X_0^i, 1 \le i \le N\}$ are i.i.d. positive r.v.s with distribution F_{X_0} and mean m_0, which are also independent of the N noise sequences.

(A2) (i) The function G: $[0, \infty) \times D_W \to [0, \infty)$ is continuous; (ii) for a fixed $w \in D_W$, $G(z, w)$ is a decreasing function of $z \in [0, \infty)$.

(A3) (iii) $EG(0, W) < \infty$ and $EG(p, W) > 0$ for each $p \in [0, \infty)$.

(A2) implies congestion effect: when the aggregate investment level increases, the production becomes less efficient.

Example. Suppose $G(z, w) = \frac{\alpha w}{1+\delta z^\eta}$, where $\alpha > 0, \delta > 0, \eta > 0$ are parameters.

How to design strategies?

- Procedures to find decentralized strategies in the mean field game.

Step 1: mean field limit

Now agent i considers the optimal control problem with dynamics

$$X_{t+1}^i = G(p_t, W_t^i)u_t^i, \qquad t \geq 0, \tag{3.2}$$

where $u_t^i \in [0, X_t^i]$. Note $G(u_t^{(N)}, W_t^i) \to G(p_t, W_t^i)$.

The utility functional is now written as

$$\bar{J}_i(u^i, (p_t)_0^{T-1}, 0) = E\sum_{t=0}^{T} \rho^t v(X_t^i - u_t^i), \tag{3.3}$$

Step 2: optimal control (for the limiting problem)

Dynamic programming equation with $t = 0, 1, \ldots, T-1$:

$$V_i(x,t) = \max_{0 \le u_i \le x} \left[v(x-u_i) + \rho E V_i(G(p_t, W_t^i)u_i,\ t+1) \right],$$

Denote $\Phi(z) = \rho E G^\gamma(z, W)$ and $\phi(z) = \Phi^{\frac{1}{\gamma-1}}(z)$.

Theorem *(i)* The value function $V_i(x,t) = \frac{1}{\gamma} D_t^{\gamma-1} x^\gamma$, where

$$D_t = \frac{\phi(p_t) D_{t+1}}{1+\phi(p_t)D_{t+1}}, \quad t \le T-1, \quad D_T = 1. \qquad (3.4)$$

(ii) The optimal control has the feedback form

$$u_t^i = \frac{X_t^i}{1+\phi(p_t)D_{t+1}}, \quad t \le T-1, \quad u_T^i = 0. \qquad (3.5)$$

Step 3: consistency

For the closed-loop system, by symmetry, $\lim_{N\to\infty} Eu_t^{(N)} = Eu_t^i =: \Lambda_t(p_0, \ldots, p_{T-1})$, which should coincide with p_t.

Define the operator $\Lambda = (\Lambda_0, \ldots, \Lambda_{T-1})$. Fixed point equation:

$$(p_0, \ldots, p_{T-1}) = \Lambda(p_0, \ldots, p_{T-1}).$$

Theorem Λ has a fixed point in a rectangle region.

Proof. Brouwer fixed point theorem.

Construct decentralized strategies

By Steps 1-3, solve $(p_t)_0^{T-1}$, and further determine $(D_t)_0^T$.

Then denote

$$\check{u}_t^i = \frac{X_t^i}{1+\phi(p_t)D_{t+1}}, \quad t \le T-1.$$

where

$$X_{t+1}^i = G(\check{u}_t^{(N)}, W_t^i)\check{u}_t^i, \ t \ge 0.$$

Question: performance of these strategies?

Step 4: ε-Nash

Theorem The set of strategies $\{\check{u}_t^i, 0 \le t \le T, 1 \le i \le N\}$ obtained from steps 1-3 is an ε_N-Nash equilibrium, i.e., for any $i \in \{1, \ldots, N\}$,

$$\sup_{u^i} J_i(u^i, \check{u}^{-i}) - \varepsilon_N \le J_i(\check{u}^i, \check{u}^{-i}) \le \sup_{u^i} J_i(u^i, \check{u}^{-i}),$$

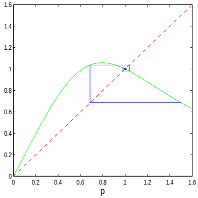
where u^i is a centralized strategy (i.e., depending on all $X_t^1, \cdots, X_t^N$) and $0 \le \varepsilon_N \to 0$ as $N \to \infty$.

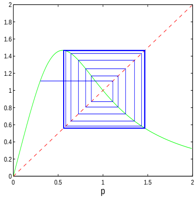
Interpretation: Global sample path based information has diminishing value!

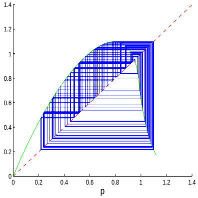
Infinite horizon and out-of-equilibrium behavior

- Formulate the infinite horizon game
- Try to solve a "stationary strategy" satisfying consistency requirement in MFG
- Slightly perturb the initial condition of the mean field system from "the steady state".
- Different situations: stable equilibrium, limit cycle, chaos.

See (Huang, DGAA'13, MFG special issue) for detail.







Continuous time modeling

Mean field production dynamics:

$$dX_t = F(m_t, X_t)dt - \delta X_t dt - C_t dt - \sigma X_t dW_t, \quad t \geq 0$$

- X_t: the capital of a representative agent, $X_0 > 0$, $EX_0 < \infty$.
- $-(\delta dt + \sigma dW_t)$: stochastic depreciation.
- $C_t \geq 0$: consumption.
- m_t: determined from the law of X_t by $m_t = EX_t$ (for simplicity); interpreted as the state average of a large number of similar agents with independent dynamics.
- $F(m, x)$: continuous function of (m, x), where $m \geq 0$, $x \geq 0$.

See next page for motivation.

Background for the previous infinite population model

A finite population of n agents.

$$dX_t^i = F(X_t^{(n)}, X_t^i)dt - \delta X_t^i dt - C_t^i dt - \sigma X_t^i dW_t^i,$$

- X_t^i: the capital of agent i; $\{X_0^i, 1 \le i \le n\}$: i.i.d. initial states
- $X_t^{(n)} = \frac{1}{n}\sum_{i=1}^n X_t^i$: the mean field coupling term
- $\{W_t^i, i = 1, \dots n\}$: i.i.d. standard Brownian motions.

For large n, we approximate $X_t^{(n)}$ by m_t and this can be heuristically justified by the law of large numbers as long as the control has certain symmetry and does not introduce correlation.

$$\Longrightarrow dX_t^i = F(m_t, X_t^i)dt - \delta X_t^i dt - C_t^i dt - \sigma X_t^i dW_t^i$$

Continuous time modeling

The utility functional:

$$J = E\left[\int_0^T e^{-\rho t} U_0(C_t)dt + e^{-\rho T} S_0(m_T, X_T)\right],$$

- ϕ $(= U_0,\ S_0(m_T, \cdot))$ is a smooth, increasing, and strictly concave function (i.e., $\phi''(z) < 0$) on $(0, \infty)$ and $\phi(0) = 0$, $\phi'(0) = \infty$, $\phi'(\infty) = 0$. Example: $\phi(C_t) = \frac{1}{\gamma} C_t^\gamma$.
- $S_0(m_T, X_T) > 0$: the terminal payoff.
- The motivation to introduce a dependence of S_0 on m_T
  - In a decision environment with congestion effect, the favor on X_T should take into account the collective behavior of others
  - It is possible to generalize $U_0(C_t) \longrightarrow U_0(EC_t, C_t)$ (need to freeze EC_t during control design)

Continuous time modeling

Assumptions:

- (A1) For each fixed x, F is a decreasing function of m.
  - (A1') Special case: When $F = A(m)x^\alpha$, $A(\cdot)$ is a continuous and strictly decreasing function on $[0, \infty)$.
- (A2) For each fixed m, F is an increasing concave function of $x \in (0, \infty)$. Furthermore, the Inada condition holds: (1) $F(m, 0) = 0$, $F_x(m, 0) = \infty$, $F_x(m, \infty) = 0$.

This concavity indicates diminishing return to scale in production. The admissible control set consists of all consumption processes C_t such that $X_t \geq 0$ for all $t \in [0, T]$.

Continuous time modeling

We write the dynamic programming equation

$$\rho V(t,x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + \sup_c \left[U_0(c) + (F(m_t,x) - \delta x - c) V_x \right], \qquad (4.1)$$

$$V(T,x) = S_0(m_T,x).$$

Under mild conditions, the equation may be interpreted in terms of certain generalized solutions (such as a viscosity solution). We proceed to simplify the above equation. Define the function

$$\psi(z) = \sup_c [U_0(c) - cz], \qquad z > 0.$$

By the concavity assumption on U_0, there is a unique maximizer

$$\hat{c}(z) = \arg\max_c [U_0(c) - cz], \qquad z > 0.$$

Continuous time modeling

The mean field game derives the solution system:

$$\rho V(t,x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + (F(m_t,x) - \delta x)V_x + \psi(V_x), \quad (4.2)$$

$$V(T,x) = S_0(m_T,x), \quad (4.3)$$

$$dX_t = F(m_t,X_t)dt - \delta X_t dt - \hat{c}(V_x(t,X_t))dt - \sigma X_t dW_t, \quad (4.4)$$

$$m_t = EX_t, \qquad \text{(consistency condition)} \quad (4.5)$$

(the third equation is a special McKean-Vlasov equation). A meaningful solution (V,m) should fulfill the requirement

$$V_x(t,x) > 0, \quad x > 0.$$

Our plan is to identify an important class of models for which more explicit computation can be developed.

Continuous time modeling: Cobb-Douglas with HARA

The dynamics:

$$dX_t = A(m_t)X_t^{\alpha}dt - \delta X_t dt - C_t dt - \sigma X_t dW_t, \qquad (4.6)$$

The utility functional:

$$J = \frac{1}{\gamma}E\left[\int_0^T e^{-\rho t}C_t^{\gamma}dt + e^{-\rho T}\eta\lambda(m_T)X_T^{\gamma}\right]. \qquad (4.7)$$

- $A(m)$ satisfies (A1'). $F(m,x) = A(m)x^{\alpha}$ is a mean field version of the Cobb-Douglas production function with capital x and a constant labor size.
- The function $\lambda > 0$ is continuous and decreasing on $[0,\infty)$.
- Take the standard choice $\gamma = 1-\alpha$ (equalizing the coefficient of the relative risk aversion to capital share)

Continuous time modeling: Cobb-Douglas with HARA

The mean field solution system reduces to

$$\rho V(t,x) = V_t + \frac{\sigma^2 x^2}{2} V_{xx} + (A(m_t)x^{1-\gamma} - \delta x)V_x + \frac{1-\gamma}{\gamma} V_x^{\frac{\gamma}{\gamma-1}},$$

$$V(T,x) = \frac{\lambda(m_T)\eta}{\gamma} x^\gamma,$$

$$dX_t = A(m_t)X_t^{1-\gamma}dt - \delta X_t dt - C_t dt - \sigma X_t dW_t,$$

$$m_t = EX_t.$$

Continuous time modeling: Cobb-Douglas with HARA

We try a solution of the form

$$V(t,x) = \frac{1}{\gamma}[p(t)x^\gamma + h(t)], \quad x > 0,\ t \geq 0.$$

We obtain two equations

$$\dot{p}(t) = \left[\rho + \frac{\sigma^2\gamma(1-\gamma)}{2} + \delta\gamma\right] p(t) - (1-\gamma)p^{\frac{\gamma}{\gamma-1}}(t) \qquad (4.8)$$

$$\dot{h}(t) = \rho h(t) - A(m_t)\gamma p(t), \qquad (4.9)$$

with the terminal conditions $p(T) = \lambda(m_T)\eta$ and $h(T) = 0$.
Proposition: For fixed m_t, The ODE system (4.8)-(4.9) has a unique solution (p, h) and the optimal control is given in the feedback form

$$C_t = p^{\frac{1}{\gamma-1}}(t)X_t.$$

Continuous time modeling: Cobb-Douglas with HARA

The solution equation system of the mean field game reduces to

$$\dot{p}(t) = \left[\rho + \frac{\sigma^2\gamma(1-\gamma)}{2} + \delta\gamma\right] p(t) - (1-\gamma)p^{\frac{\gamma}{\gamma-1}}(t)$$

$$\dot{h}(t) = \rho h(t) - A(m_t)\gamma p(t),$$

$$dZ_t = \left\{\gamma A(m_t) - \left[\gamma\delta - \gamma\varphi^{-1}(t) - \frac{\sigma^2\gamma(1-\gamma)}{2}\right] Z_t\right\} dt - \gamma\sigma Z_t dW_t,$$

$$m_t = EZ_t^{\frac{1}{\gamma}} \quad (= EX_t),$$

where $p(T) = \lambda(m_T)\eta$ and $h(T) = 0$. $\varphi(t)$ can be explicitly determined by $\lambda(m_T)$ and other constant parameters.

- Existence = fixed point problem. Fix m_t; uniquely solve p, h; further solve $Z_t(m(\cdot))$. Then $m_t = EZ_t^{\frac{1}{\gamma}}(m(\cdot))$.

Concluding remarks

Computation:

- Except LQG (Huang et. al. 2003, 2007; Li et. al. 08; Bardi, 2012, ...), LQEG (Tembine et. al., 2011) cases, closed-form solutions for mean field games are rare
- HARA utility is useful to develop explicit computations

Mean field game literature

Related literature: mean field games (only a partial list)

- J.M. Lasry and P.L. Lions (2006a,b, JJM'07): Mean field equilibrium; O. Gueant (JMPA'09); GLL'11 (Springer): Human capital optimization
- G.Y. Weintraub et. el. (NIPS'05, Econometrica'08): Oblivious equilibria for Markov perfect industry dynamics; S. Adlakha, R. Johari, G. Weibtraub, A. Goldsmith (CDC'08): further generalizations with OEs
- M. Huang, P.E. Caines and R.P. Malhame (CDC'03, 04, CIS'06, TAC'07): Decentralized ε-Nash equilibrium in mean field dynamic games; M. Nourian, Caines, et. al. (TAC'12): collective motion and adaptation; A. Kizilkale and P. E. Caines (Preprint'12): adaptive mean field LQG games
- T. Li and J.-F. Zhang (IEEE TAC'08): Mean field LQG games with long run average cost; M. Bardi (Net. Heter. Media'12) LQG
- H. Tembine et. al. (GameNets'09): Mean field MDP and team; H. Tembine, Q. Zhu, T. Basar (IFAC'11): Risk sensitive mean field games

Related literature (ctn)

- A. Bensoussan et. al. (2011, 2012, Preprints) Mean field LQG games (and nonlinear diffusion models).
- H. Yin, P.G. Mehta, S.P. Meyn, U.V. Shanbhag (IEEE TAC'12): Nonlinear oscillator games and phase transition; Yang et. al. (ACC'11); Pequito, Aguiar, Sinopoli, Gomes (NetGCOOP'11): application to filtering/estimation
- D. Gomes, J. Mohr, Q. Souza (JMPA'10): Finite state space models
- V. Kolokoltsov, W. Yang, J. Li (preprint'11): Nonlinear markov processes and mean field games

Related literature (ctn)

- Z. Ma, D. Callaway, I. Hiskens (IEEE CST'13): recharging control of large populations of electric vehicles
- Y. Achdou and I. Capuzzo-Dolcetta (SIAM Numer.'11): Numerical solutions to mean field game equations (coupled PDEs)
- R. Buckdahn, P. Cardaliaguet, M. Quincampoix (DGA'11): Survey
- R. Carmona and F. Delarue (Preprint'12): McKean-Vlasov dynamics for players, and probabilistic approach
- R. E. Lucas Jr and B. Moll (Preprint'11): Economic growth (a trade-off for individuals to allocate time for producing and acquiring new knowledge)
- Huang (2010); Nguyen and Huang (2012); Nourian and Caines (2012); Bensoussan et al (2013): Major player models.

Related literature (ctn):

Mean field type optimal control:

- D. Andersson and B. Djehiche (AMO'11): Stochastic maximum principle
- J. Yong (Preprint'11): control of mean field Volterra integral equations
- T. Meyer-Brandis, B. Oksendal and X. Y. Zhou (2012): SMP.

There is a single decision maker who has significant influence on the mean of the underlying state process.

A player in a mean field game (except major player models) has little impact on the mean field.