Variational approach to mean field games with density constraints

Alpár Richárd Mészáros

LMO, Université Paris-Sud

(based on ongoing joint works with
F. Santambrogio, P. Cardaliaguet and F. J. Silva)

CONFERENCE ON OPTIMIZATION, TRANSPORTATION AND EQUILIBRIUM IN ECONOMICS, SEPT. 15-19, 2014, TORONTO

The content of the talk

1. Basic models of Mean Field Games, after J.-M. Lasry and P.-L. Lions
2. Variational approaches for MFGs
3. A link with Optimal Transport and its Benamou-Brenier formulation
4. Second order stationary MFGs
5. Treating density constraints and penalizations for second order stationary MFGs
6. Summary, works in progress and open questions

A short history of Mean Field Games

- This theory was introduced recently (in 2006-2007) by J.-M. Lasry and P.-L. Lions[1,2] [3] in a series of papers and a series of lectures by P.-L. Lions at Collège de France;

---

[1] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen I. Le cas stationnaire, *C. R. Math. Acad. Sci. Paris*, (2006).

[2] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen II. Horizon fini et contrôle optimal, *C. R. Math. Acad. Sci. Paris*, (2006).

[3] Lasry, J.-M., Lions, P.-L. Mean field games, *Jpn. J. Math.*, (2007).

A short history of Mean Field Games

- This theory was introduced recently (in 2006-2007) by J.-M. Lasry and P.-L. Lions[1][2] [3] in a series of papers and a series of lectures by P.-L. Lions at Collège de France;
- Analysis of differential games with a very large number of “small” players (agents);

[1]Lasry, J.-M., Lions, P.-L. Jeux à champ moyen I. Le cas stationnaire, *C. R. Math. Acad. Sci. Paris*, (2006).

[2]Lasry, J.-M., Lions, P.-L. Jeux à champ moyen II. Horizon fini et contrôle optimal, *C. R. Math. Acad. Sci. Paris*, (2006).

[3]Lasry, J.-M., Lions, P.-L. Mean field games, *Jpn. J. Math.*, (2007).

A short history of Mean Field Games

- This theory was introduced recently (in 2006-2007) by J.-M. Lasry and P.-L. Lions[1][2] [3] in a series of papers and a series of lectures by P.-L. Lions at Collège de France;
- Analysis of differential games with a very large number of “small” players (agents);
- The models are derived from a “continuum limit”, letting the number of agents go to infinity (similarly to mean field limit in Statistical Mechanics and Physics $\rightarrow$ Bolzmann or Vlasov equations)

[1] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen I. Le cas stationnaire, *C. R. Math. Acad. Sci. Paris*, (2006).

[2] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen II. Horizon fini et contrôle optimal, *C. R. Math. Acad. Sci. Paris*, (2006).

[3] Lasry, J.-M., Lions, P.-L. Mean field games, *Jpn. J. Math.*, (2007).

A short history of Mean Field Games

- This theory was introduced recently (in 2006-2007) by J.-M. Lasry and P.-L. Lions[1] [2] [3] in a series of papers and a series of lectures by P.-L. Lions at Collège de France;
- Analysis of differential games with a very large number of "small" players (agents);
- The models are derived from a "continuum limit", letting the number of agents go to infinity (similarly to mean field limit in Statistical Mechanics and Physics $\rightarrow$ Bolzmann or Vlasov equations)
- Real life applications in Economy, Finance and Social Sciences

[1] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen I. Le cas stationnaire, *C. R. Math. Acad. Sci. Paris*, (2006).

[2] Lasry, J.-M., Lions, P.-L. Jeux à champ moyen II. Horizon fini et contrôle optimal, *C. R. Math. Acad. Sci. Paris*, (2006).

[3] Lasry, J.-M., Lions, P.-L. Mean field games, *Jpn. J. Math.*, (2007).

A typical model for second order MFG

$$
\begin{cases}
(i) & -\partial_t u + \nu\Delta u + H(x, m, \nabla u) = f(x, m) & \text{in } (0, T) \times \mathbb{R}^d \\
(ii) & \partial_t m - \nu\Delta m - \nabla \cdot (\nabla_p H(x, m, \nabla u)m) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\
(iii) & m(0) = m_0, u(T, x) = G(x, m(T)) & \text{in } \mathbb{R}^d.
\end{cases}
\tag{1}
$$

A typical model for second order MFG

$$
\begin{cases}
(i) & -\partial_t u + \nu \Delta u + H(x, m, \nabla u) = f(x, m) & \text{in } (0, T) \times \mathbb{R}^d \\
(ii) & \partial_t m - \nu \Delta m - \nabla \cdot (\nabla_p H(x, m, \nabla u) m) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\
(iii) & m(0) = m_0, u(T, x) = G(x, m(T)) & \text{in } \mathbb{R}^d.
\end{cases}
\tag{1}
$$

- Assumptions: $\nu \geq 0$ is a parameter; the Hamiltonian H is convex in its last variable; m_0 (and $m(t)$) is the density of a probability measure;

A typical model for second order MFG

$$\begin{cases} (i) & -\partial_t u + \nu\Delta u + H(x, m, \nabla u) = f(x, m) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m - \nu\Delta m - \nabla\cdot(\nabla_p H(x, m, \nabla u)m) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{1}$$

- Assumptions: $\nu \geq 0$ is a parameter; the Hamiltonian H is convex in its last variable; m_0 (and $m(t)$) is the density of a probability measure;
- u is the value function of an arbitrary agent, m is the distribution of the agents;

A heustistical interpretation

- An arbitrary agent controls the stochastic differential equation

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + \sqrt{2\nu}\mathrm{d}B_t,$$

where B_t is a standard Brownian motion.

A heustistical interpretation

- An arbitrary agent controls the stochastic differential equation

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + \sqrt{2\nu}\mathrm{d}B_t,$$

where B_t is a standard Brownian motion.

- He aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T L(X_s, m(s), \alpha_s) + f(X_s, m(s))\mathrm{d}s + G(X_T, m(T))\right],$$

where L is the usual Legendre-Flenchel conjugate of H w.r.t. the p variable.

A heustistical interpretation

- An arbitrary agent controls the stochastic differential equation

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + \sqrt{2\nu}\mathrm{d}B_t,$$

where B_t is a standard Brownian motion.

- He aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T L(X_s, m(s), \alpha_s) + f(X_s, m(s))\mathrm{d}s + G(X_T, m(T))\right],$$

where L is the usual Legendre-Flenchel conjugate of H w.r.t. the p variable.

- His optimal control is (at least heuristically) given in feedback form by $\alpha^*(t,x) = -\nabla_p H(x, m, \nabla u)$.

A typical model of first order MFG system

- A typical model for a first order (deterministic) MFG system is the following:

A typical model of first order MFG system

- A typical model for a first order (deterministic) MFG system is the following:

$$\begin{cases} (i) & -\partial_t u(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = f(x, m(t)) \quad \text{in } (0,T) \times \mathbb{R}^d \\ (ii) & \partial_t m(t,x) - \nabla \cdot (\nabla u(t,x) m(t,x)) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x, m(T)) \quad \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

A typical model of first order MFG system

- A typical model for a first order (deterministic) MFG system is the following:

$$\begin{cases} (i) & -\partial_t u(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = f(x, m(t)) \quad \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m(t,x) - \nabla\cdot(\nabla u(t,x) m(t,x)) = 0 \quad \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x, m(T)) \quad \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

- u corresponds to the value function of a typical agent who controls his velocity $\alpha(t)$ and has to minimize his cost

$$\int_0^T \left(\frac{1}{2}|\alpha(t)|^2 + f(x(t), m(t))\right) \mathrm{d}t + G(x(T), m(T)),$$

where $x'(s) = \alpha(s)$ and $x(0) = x_0$.

A typical model of first order MFG system

- A typical model for a first order (deterministic) MFG system is the following:

$$\begin{cases} (i) & -\partial_t u(t,x) + \frac{1}{2}|\nabla u(t,x)|^2 = f(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m(t,x) - \nabla\cdot(\nabla u(t,x) m(t,x)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & m(0) = m_0, u(T,x) = G(x, m(T)) & \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

- u corresponds to the value function of a typical agent who controls his velocity $\alpha(t)$ and has to minimize his cost

$$\int_0^T \left(\frac{1}{2}|\alpha(t)|^2 + f(x(t), m(t))\right) \mathrm{d}t + G(x(T), m(T)),$$

where $x'(s) = \alpha(s)$ and $x(0) = x_0$.

- The distribution of the other agents is represented by the density $m(t)$. Then their "feedback strategy" is given by $\alpha(t,x) = -\nabla u(t,x)$.

The variational formulation

- We can obtain the solution of the system (2) by minimization of a global functional:

$$\min \int_0^T \int_\Omega \left(\frac{1}{2} |\alpha(t,x)|^2 \rho(t,x) + F(\rho(t,x)) \right) \mathrm{d}x\mathrm{d}t - \int_\Omega G(x)\rho(T,x)\,\mathrm{d}x,$$

among solutions (ρ, α) of the continuity equation $\partial_t \rho + \nabla \cdot (\rho\alpha) = 0$ with the initial datum $\rho(0,x) = \rho_0(x)$.

[4]P. Cardaliaguet, Week solutions for first order mean field games with local coupling, *preprint*

[5]P. Cardaliaguet, P.J. Graber, Mean field games systems of first order, *preprint*

The variational formulation

- We can obtain the solution of the system (2) by minimization of a global functional:

$$\min \int_0^T \int_\Omega \left(\frac{1}{2}|\alpha(t,x)|^2 \rho(t,x) + F(\rho(t,x)) \right) \mathrm{d}x\mathrm{d}t - \int_\Omega G(x)\rho(T,x)\,\mathrm{d}x,$$

among solutions (ρ, α) of the continuity equation $\partial_t \rho + \nabla \cdot (\rho\alpha) = 0$ with the initial datum $\rho(0,x) = \rho_0(x)$.

- Recently many interesting results in this variational direction, see for instance the works of P. Cardaliaguet and P. J. Graber[4], [5]

[4] P. Cardaliaguet, Week solutions for first order mean field games with local coupling, *preprint*

[5] P. Cardaliaguet, P.J. Graber, Mean field games systems of first order, *preprint*

The variational formulation

- We can obtain the solution of the system (2) by minimization of a global functional:

$$\min \int_0^T \int_\Omega \left(\frac{1}{2}|\alpha(t,x)|^2 \rho(t,x) + F(\rho(t,x)) \right) \mathrm{d}x\mathrm{d}t - \int_\Omega G(x)\rho(T,x)\,\mathrm{d}x,$$

among solutions (ρ, α) of the continuity equation $\partial_t \rho + \nabla \cdot (\rho\alpha) = 0$ with the initial datum $\rho(0,x) = \rho_0(x)$.

- Recently many interesting results in this variational direction, see for instance the works of P. Cardaliaguet and P. J. Graber[4], [5]
- $F(x,\cdot)' = f(x,\cdot)$ and it is convex. The above functional recalls the functional studied by Benamou and Brenier to give a dynamical formulation of optimal transport.

[4] P. Cardaliaguet, Week solutions for first order mean field games with local coupling, *preprint*

[5] P. Cardaliaguet, P.J. Graber, Mean field games systems of first order, *preprint*

Optimal transportation and its Benamou-Brenier formulation

- For two (regular enough) probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ we define

$$W_2^2(\mu,\nu) := \inf\left\{\int_\Omega \frac{1}{2}|x - T(x)|^2 \,\mathrm{d}\mu : T : \Omega \to \Omega,\ T_{\#}\mu = \nu\right\}$$

- Teorem [Y. Brenier, '87]: Under suitable assumptions there exists T (optimal transport map) which is a gradient of a convex function.
- We can solve this problem via a dynamic formulation due to J.-D. Benamou and Y. Brenier, '00:

$$\min_\alpha \left\{\int_0^1 \int_\Omega \frac{1}{2}|\alpha_t|^2 \,\mathrm{d}\rho_t \mathrm{d}t :\ \partial_t \rho_t + \nabla\cdot(\rho_t\alpha_t) = 0, \rho_0 = \mu, \rho_1 = \nu\right\}.$$

- W_2 metrizes the weak-* topology on $\mathcal{P}(\Omega)$ for compact domains Ω.

The case of density penalization

- To treat density constraints for MFGs we consider the case $f(\rho) = \rho^{n-1} (n > 1)$ (in this case $F(\rho) = \frac{1}{n}\rho^n$)

The case of density penalization

- To treat density constraints for MFGs we consider the case $f(\rho) = \rho^{n-1} (n > 1)$ (in this case $F(\rho) = \frac{1}{n}\rho^n$)
- Now the question is what happens in the above minimization problem as $n \to \infty$.

The case of density penalization

- To treat density constraints for MFGs we consider the case $f(\rho) = \rho^{n-1} (n > 1)$ (in this case $F(\rho) = \frac{1}{n}\rho^n$)
- Now the question is what happens in the above minimization problem as $n \to \infty$.
- We will have in the limit

$$\min \int_0^T \int_\Omega \frac{1}{2}|\alpha(t,x)|^2 \rho(t,x)\,\mathrm{d}x\mathrm{d}t - \int_\Omega G(x)\rho(T,x)\mathrm{d}x,$$

with the same assumptions as above and with the additional assumption that $\rho(t,x) \leq 1$ a.e.

The case of density penalization

- To treat density constraints for MFGs we consider the case $f(\rho) = \rho^{n-1} (n > 1)$ (in this case $F(\rho) = \frac{1}{n}\rho^n$)
- Now the question is what happens in the above minimization problem as $n \to \infty$.
- We will have in the limit

$$\min \int_0^T \int_\Omega \frac{1}{2}|\alpha(t,x)|^2 \rho(t,x)\,\mathrm{d}x\mathrm{d}t - \int_\Omega G(x)\rho(T,x)\mathrm{d}x,$$

  with the same assumptions as above and with the additional assumption that $\rho(t,x) \leq 1$ a.e.
- However it is not not that clear how to define the equilibria and what are the optimality conditions (the MFG system) in the limit $\to$ subject of an ongoing work with P. Cardaliaguet and F. Santambrogio.

Stationary second order MFGs

- Stationary MFG systems could be seen as long time average of time-dependent ones (see the original papers of Lasry and Lions and [6])

[6] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, Long time average of mean field games with a nonlocal coupling, *SIAM J. Contr. Optim.*, (2013).

Stationary second order MFGs

- Stationary MFG systems could be seen as long time average of time-dependent ones (see the original papers of Lasry and Lions and [6])
- One can prove the convergence (in a reasonable sense as $T \to +\infty$) of the solutions of the system

$$
\begin{cases}
-\partial_t u^T - \Delta u^T + \frac{1}{2}\left|\nabla u^T\right|^2 = f(x, m^T), \\
\partial_t m^T - \Delta m^T - \nabla \cdot (m^T \nabla u^T) = 0, \\
m^T(0) = m_0, \; u^T(T) = G
\end{cases}
$$

to the solutions of the ergodic system

$$
\begin{cases}
\lambda - \Delta u + \frac{1}{2}\left|\nabla u\right|^2 = f(x, m), \\
-\Delta m - \nabla \cdot (m \nabla u) = 0, \\
\int_\Omega m = 1, \; \int_\Omega u = 0.
\end{cases}
$$

[6] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, Long time average of mean field games with a nonlocal coupling, *SIAM J. Contr. Optim.*, (2013).

Treating directly the density constraint

- The ergodic problem corresponds to the optimality conditions of the following optimization problem in a bounded open set $\Omega \subset \mathbb{R}^d$

$$\min_{(m,w)} \mathcal{L}_2(m,w) + \mathcal{F}(m), \qquad (P_2)$$

subject to $-\Delta m + \nabla \cdot w = 0$ with Neumann b.c. $(\nabla m - w) \cdot n = 0$ on $\partial\Omega$ and $\int_\Omega m = 1$.

Treating directly the density constraint

- The ergodic problem corresponds to the optimality conditions of the following optimization problem in a bounded open set $\Omega \subset \mathbb{R}^d$

$$\min_{(m,w)} \mathcal{L}_2(m,w) + \mathcal{F}(m), \qquad (P_2)$$

subject to $-\Delta m + \nabla \cdot w = 0$ with Neumann b.c. $(\nabla m - w)\cdot n = 0$ on $\partial\Omega$ and $\int_\Omega m = 1$. Here we define for $q > 1$, $\ell_q : \mathbb{R} \times \mathbb{R}^d \to \overline{\mathbb{R}}$,

$$\ell_q(a,b) := \begin{cases} \frac{1}{q}\frac{|b|^q}{a^{q-1}}, & \text{if } a > 0, \\ 0, & \text{if } (a,b) = (0,0), \\ +\infty, & \text{otherwise.} \end{cases}$$

Treating directly the density constraint

- The ergodic problem corresponds to the optimality conditions of the following optimization problem in a bounded open set $\Omega \subset \mathbb{R}^d$

$$\min_{(m,w)} \mathcal{L}_2(m,w) + \mathcal{F}(m), \qquad (P_2)$$

subject to $-\Delta m + \nabla \cdot w = 0$ with Neumann b.c. $(\nabla m - w) \cdot n = 0$ on $\partial\Omega$ and $\int_\Omega m = 1$. Here we define for $q > 1$, $\ell_q : \mathbb{R} \times \mathbb{R}^d \to \overline{\mathbb{R}}$,

$$\ell_q(a,b) := \begin{cases} \frac{1}{q}\frac{|b|^q}{a^{q-1}}, & \text{if } a > 0, \\ 0, & \text{if } (a,b) = (0,0), \\ +\infty, & \text{otherwise.} \end{cases}$$

- We use the notation $\mathcal{F}(m) := \int_\Omega F(x, m(x))\,dx$ and $\mathcal{L}_q(m,w) := \int_\Omega \ell_q(m,w)\,dx$.
- Our objective is to study (P_q) + "$m \leq 1$" $\to$ joint work with F. J. Silva.

Interior point condition and subdifferentiability

- We work in the space $w \in L^q(\Omega)^d$, for a $q > d$, hence by a Calderón-Zygmund-type argument we obtain $m \in W^{1,q}(\Omega)$.

Interior point condition and subdifferentiability

- We work in the space $w \in L^q(\Omega)^d$, for a $q > d$, hence by a Calderón-Zygmund-type argument we obtain $m \in W^{1,q}(\Omega)$.
- As $q > d$, we have $W^{1,q}(\Omega) \hookrightarrow C^{0,1-\frac{d}{q}}(\overline{\Omega})$, $\to$ good topology to ensure the interior point condition for the constraint $m \leq 1$.

Interior point condition and subdifferentiability

- We work in the space $w \in L^q(\Omega)^d$, for a $q > d$, hence by a Calderón-Zygmund-type argument we obtain $m \in W^{1,q}(\Omega)$.
- As $q > d$, we have $W^{1,q}(\Omega) \hookrightarrow C^{0,1-\frac{d}{q}}(\overline{\Omega})$, $\to$ good topology to ensure the interior point condition for the constraint $m \leq 1$.

Proposition

Denote $E_0^m := \overline{\{m = 0\} \cap \{w = 0\}}$ and $E_1^m = \{m > 0\}$. Let us set $v := (w/m)\mathbb{1}_{\{m>0\}}$. Then, if $v \notin L^q(\Omega)$, we have that $\partial\mathcal{L}_q(m, w) = \emptyset$. Otherwise we have that $\mathcal{L}_q$ is subdifferentiable at (m, w) and

$$\partial\mathcal{L}_q(m, w) = \left\{(\alpha, \beta) \in \overline{\mathcal{A}} \ : \ \mathrm{spt}(\alpha^s) \subseteq E_0^m, \ \ \alpha^{ac} \llcorner E_1^m = -\frac{1}{q'}|v|^q \ \ \text{and} \ \ \beta \llcorner E_1^m = |v|^{q-2}v\right\},$$

where

$$\overline{\mathcal{A}} = \left\{(\alpha, \beta) \in \mathfrak{M}(\Omega) \times L^{q'}(\Omega)^d \ ; \ \alpha + \frac{1}{q'}|\beta|^{q'} \leq 0\right\},$$

and $\alpha = \alpha^{ac} + \alpha^s$ is the Lebesgue decomposition of the measure α.

Existence and optimality conditions

Theorem

The problem (P_q) *has a solution* $(m, w) \in W^{1,q}(\Omega) \times L^q(\Omega)^d$.

Existence and optimality conditions

Theorem

The problem (P_q) *has a solution* $(m, w) \in W^{1,q}(\Omega) \times L^q(\Omega)^d$.

Theorem

Let (m, w) *be a solution of problem* (P_q)*. Then,* $v := (w/m)\mathbb{1}_{\{m>0\}} \in L^q(\Omega)^d$ *and there exist* $(u, p, \lambda) \in W^{1,q'}_{\diamond}(\Omega) \times \mathfrak{M}_+(\Omega) \times \mathbb{R}$ *and* $(\alpha, \beta) \in \overline{\mathcal{A}}$ *such that* $-\Delta u \in \mathfrak{M}(\Omega)$ *and the following optimality conditions hold true*

$$\left\{ \begin{array}{rcl} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} - p - \lambda - \alpha & = & f(m), \\ \beta - \nabla u & = & 0, \\ -\Delta m + \nabla \cdot w & = & 0, \\ \mathrm{spt}(p) & \subseteq & \{m = 1\}, \\ +\text{Neumann b.c.} & \text{on} & \partial\Omega \end{array} \right. \tag{3}$$

where the first equality holds in $\mathfrak{M}(\Omega)$.

Regularity of the solutions

Corollary

Let (m, w, u, p, λ) be as in the previous theorem. Then,

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} - p - \lambda & = & f(m), \\ \nabla u \cdot n = 0 & & \text{on } \partial E_1^m \\ -\Delta m - \nabla \cdot (m|\nabla u|^{\frac{2-q}{q-1}} \nabla u) & = & 0, \\ \nabla m \cdot n = 0 & & \text{on } \partial E_1^m \end{cases} \tag{4}$$

where the first equality is satisfied in the sense of measures while the second one in the sense of distributions over E_1^m. In particular if $f \in C^\infty(\overline{\Omega})$, setting $E_2^m := \{0 < m < 1\}$, we have that $(m, u) \in C^\infty(E_2^m) \times C^\infty(E_2^m)$ is a classical solution of

$$\begin{cases} -\Delta u + \frac{1}{q'}|\nabla u|^{q'} - \lambda & = & f(m), \\ \nabla u \cdot n = 0 & & \text{on } \partial E_2^m \\ -\Delta m - \nabla \cdot (m|\nabla u|^{\frac{2-q}{q-1}} \nabla u) & = & 0, \\ \nabla m \cdot n = 0 & & \text{on } \partial E_2^m \end{cases} \tag{5}$$

The dual problem

Proposition

The dual problem of (P_q) has at least one solution and can be written as

$$-\min_{(u,p,\lambda,a)\in\mathcal{K}_D}\left\{\int_\Omega F^*(x,a)\,\mathrm{d}x+\lambda+\sigma_{\mathcal{C}}(p)\right\} \qquad (PD_q)$$

where

$$\mathcal{K}_D := \left\{(u,p,\lambda,a)\in W^{1,q'}_{\diamond}(\Omega)\times\mathfrak{M}_+(\Omega)\times\mathbb{R}\times\mathfrak{M}_{ac}(\Omega):\ -\Delta u+\frac{1}{q'}|\nabla u|^{q'}-p-\lambda\le a\right\},$$

$\mathcal{C} := \{y\in C(\overline{\Omega}) : y\le 1\}$, $\sigma_{\mathcal{C}}(p) := \sup_{y\in\mathcal{C}}\langle p,y\rangle_{\mathfrak{M}(\overline{\Omega}),C(\overline{\Omega})}$ and the PDE in the definition of the set has to be understood in the sense of measures.

The dual problem

Proposition

The dual problem of (P_q) has at least one solution and can be written as

$$-\min_{(u,p,\lambda,a)\in\mathcal{K}_D}\left\{\int_\Omega F^*(x,a)\,\mathrm{d}x+\lambda+\sigma_{\mathcal{C}}(p)\right\} \qquad (PD_q)$$

where

$$\mathcal{K}_D:=\left\{(u,p,\lambda,a)\in W^{1,q'}_\diamond(\Omega)\times\mathfrak{M}_+(\Omega)\times\mathbb{R}\times\mathfrak{M}_{ac}(\Omega):\ -\Delta u+\frac{1}{q'}|\nabla u|^{q'}-p-\lambda\le a\right\},$$

$\mathcal{C}:=\{y\in C(\overline{\Omega}):y\le 1\}$, $\sigma_{\mathcal{C}}(p):=\sup_{y\in\mathcal{C}}\langle p,y\rangle_{\mathfrak{M}(\overline{\Omega}),C(\overline{\Omega})}$ and the PDE in the definition of the set has to be understood in the sense of measures.

$\rightarrow$ Moreover $(P_q)=(PD_q)$ from where we obtain an alternative way to derive first order optimality conditions.

An approximation argument for less regular cases

- Let us consider in the problem (P_q) a cost $\mathcal{L}_q$ for a $q \leq d$. By this the interior point condition for the constraint $m \leq 1$ will be destroyed.

An approximation argument for less regular cases

- Let us consider in the problem (P_q) a cost $\mathcal{L}_q$ for a $q \leq d$. By this the interior point condition for the constraint $m \leq 1$ will be destroyed.
- Indeed, for instance for $q = 2$ and $d \geq 2$ the natural space for w is $L^2(\Omega)^d$, hence the maximal regularity for m is $H^1(\Omega)$.

An approximation argument for less regular cases

- Let us consider in the problem (P_q) a cost $\mathcal{L}_q$ for a $q \leq d$. By this the interior point condition for the constraint $m \leq 1$ will be destroyed.
- Indeed, for instance for $q = 2$ and $d \geq 2$ the natural space for w is $L^2(\Omega)^d$, hence the maximal regularity for m is $H^1(\Omega)$.$\longrightarrow$ no interior point for $m \leq 1$.

An approximation argument for less regular cases

- Let us consider in the problem (P_q) a cost $\mathcal{L}_q$ for a $q \leq d$. By this the interior point condition for the constraint $m \leq 1$ will be destroyed.
- Indeed, for instance for $q = 2$ and $d \geq 2$ the natural space for w is $L^2(\Omega)^d$, hence the maximal regularity for m is $H^1(\Omega)$.$\rightarrow$ no interior point for $m \leq 1$.
- A natural procedure to solve this issue: use an approximation of type $\mathcal{L}_2 + \varepsilon\mathcal{L}_q$ for $\mathcal{L}_2$, where $q > d$ and take the limit as $\varepsilon \to 0$.

An approximation argument for less regular cases

- Let us consider in the problem (P_q) a cost $\mathcal{L}_q$ for a $q \leq d$. By this the interior point condition for the constraint $m \leq 1$ will be destroyed.
- Indeed, for instance for $q = 2$ and $d \geq 2$ the natural space for w is $L^2(\Omega)^d$, hence the maximal regularity for m is $H^1(\Omega)$. $\to$ no interior point for $m \leq 1$.
- A natural procedure to solve this issue: use an approximation of type $\mathcal{L}_2 + \varepsilon\mathcal{L}_q$ for $\mathcal{L}_2$, where $q > d$ and take the limit as $\varepsilon \to 0$.
- For this we proved uniform (in ε) estimates for each term. We have
$$|\lambda_\varepsilon| + \|p_\varepsilon\|_{\mathfrak{M}} + \|m_\varepsilon\|_{H^1} + \|H_\varepsilon(\nabla u_\varepsilon)\|_{L^1} \leq C.$$
- This implies in particular that $\|\Delta u_\varepsilon\|_{\mathfrak{M}} \leq C$, which is enough to use again a Calderón-Zygmund-type argument and get compactness for ∇u_ε.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.
- For less regular Hamiltonians we propose a possible approach by approximation.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.
- For less regular Hamiltonians we propose a possible approach by approximation.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.
- For less regular Hamiltonians we propose a possible approach by approximation.

Work in progress, future work and open questions:

- Work in progress: similar analysis for the time dependent systems (both first and second order cases) with density constraints.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.
- For less regular Hamiltonians we propose a possible approach by approximation.

Work in progress, future work and open questions:

- Work in progress: similar analysis for the time dependent systems (both first and second order cases) with density constraints.
- Future work: study the long time average of time dependent systems with density constraints.

Summary, perspectives and open problems

Summary

- We established existence results and characterization of the solutions of stationary second order MFG systems with density constraints.
- For regular enough Hamiltonians we could use some results from optimal control theory for mixed PDE and state constraint.
- For less regular Hamiltonians we propose a possible approach by approximation.

Work in progress, future work and open questions:

- Work in progress: similar analysis for the time dependent systems (both first and second order cases) with density constraints.
- Future work: study the long time average of time dependent systems with density constraints.
- Open question: find a good notion of equilibria.

Thank you for your attention!