Uniqueness of the compactly supported weak solution to the relativistic Vlasov-Darwin system

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Relativistic Vlasov-Darwin system

- The relativistic Vlasov-Darwin (RVD) system is a kinetic equation that describes the evolution of a collisionless plasma whose (charged) particles interact through their self-induced electromagnetic field and move at a speed "not too fast" compared with the speed of light.
- It is obtained from the relativistic Vlasov-Maxwell (RVM) system by neglecting the transversal part of the displacement current (i.e. the time derivative of the electric field) in Maxwell-Ampère's equation.
- RVD system approximates RVM system at the rate $\mathcal{O}(c^{-3})$, where c is the speed of light.
- **Goal:** Prove uniqueness of weak solutions to RVD system under the assumption that the solutions remain compactly supported at all times.

Relativistic Vlasov equation

The RVM system is the coupling of the relativistic Vlasov (RV) equation and Maxwell's (M) equations.

- **Relativistic Vlasov equation**. Consider an ensemble of identical charged particles with mass m and charge q, (*normalize* $m = q = 1$). Denote by $f(t, x, \xi)$ the density of particles at time $t \geq 0$ in the phase space $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$.

$$(\textbf{RV}): \qquad \partial_t f + v(\xi) \cdot \nabla_x f + (E + c^{-1} v(\xi) \times B) \cdot \nabla_\xi f = 0$$

$$v = \frac{\xi}{\sqrt{1 + c^{-2}|\xi|^2}} \equiv \text{relativistic velocity; } c \equiv \text{speed of light.}$$

$E = E(t, x)$ and $B = B(t, x)$ are the electric and magnetic fields induced by the particles. They satisfy Maxwell's equations.

Maxwell's equations

$$(\mathbf{M}) : \quad \begin{cases} \nabla \times B - c^{-1}\partial_t E = 4\pi c^{-1} j & ; \quad \nabla \cdot B = 0 \\ \nabla \times E + c^{-1}\partial_t B = 0 & ; \quad \nabla \cdot E = 4\pi\rho \end{cases}$$

where

$\rho = \rho(t,x)$ is the charge density,

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)d\xi, \quad (\textit{normalize } q \equiv 1),$$

$j = j(t,x)$ is the current density,

$$j(t,x) = \int_{\mathbb{R}^3} v(\xi) f(t,x,\xi)d\xi,$$

related by $\partial_t \rho + \nabla_x \cdot j = 0$ (charge conservation law).

Darwin's equations

- **Helmholtz decomposition:**

$$E = E_L + E_T \quad \text{where} \quad \nabla \times E_L = 0, \quad \nabla \cdot E_T = 0.$$

If we neglect $c^{-1}\partial_t E_T$ in Maxwell-Ampère's law, then Maxwell's equations become **Darwin's equations**:

$$(\mathbf{D}) : \quad \begin{cases} \nabla \times B - c^{-1}\partial_t E_L = 4\pi c^{-1} j & ; \quad \nabla \cdot B = 0 \\ \nabla \times E_T + c^{-1}\partial_t B = 0 & ; \quad \nabla \cdot E_L = 4\pi\rho \end{cases}$$

- The **RVD system** is the coupling of the relativistic Vlasov equation (RV) and Darwin's equations (D).
- Physically, Darwin's approximation makes sense when the evolution of the electromagnetic field is "slower" than the speed of light (*at the order* $\mathcal{O}(c^{-3})$).

Approximations of RVM system

Viewing c as a parameter, and let $c \to \infty$, it is shown that for,

- **VD system** (*[Bauer & Kunze, '05]*). If (f, E, B) and (f^D, E^D, B^D) are (classical) solutions to the RVM and RVD with the same (compactly supported) initial data f_0 on some interval $[0, T)$, then

$$|f - f^D| + |E - E^D| + |B - B^D| \leq Mc^{-3}$$

  for all $(x, \xi, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$; M is indepedent of c.

- **Vlasov-Poisson system** (*[Schaeffer, '86]*). Also if (f^P, E^P) solves the Valsov-Poisson system with initial data f_0, then

$$\partial_t f + \xi \cdot \nabla_x f + E \cdot \nabla_\xi f = 0, \quad f(t=0) = f_0, \quad \text{then},$$

$$|f - f^P| + |E - E^P| + |B| \leq Mc^{-1}$$

- The **Vlasov-Poisson system** is a 1st order approximation of the RVM system (*called the classical limit of RVM*), which only takes into account the electric field induced by the particles, (*the magnetic field is neglected*). This model gives a 'poor' approximation of the RVM system when the effect of the magnetic field is significant.
- In contrast, the **RVD system**, which is a 3rd order approximation of the RVM system (*called the quasi-static limit*), preserves the fully couple electromagnetic fields induced by the particles. This is a more desirable model for numerical simulations of collisionless plasma.
- Yet, as the Vlasov-Poisson system, the RVD system has an elliptic structure, while the full RVM system is of hyperbolic type.

Potential formulation of (M)

- Since $\nabla \cdot B = 0$, by Helmholtz decomposition: there exists a vector potential $A = A(t, x)$ s.t.

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

A is not uniquely defined; any $A' = A + \nabla\psi$ is acceptable.

- Insert $B = \nabla \times A$ into $\nabla \times E + c^{-1}\partial_t B = 0$ implies that: there exists a scalar potential $\Phi = \Phi(t, x)$ s.t.

$$E + c^{-1}\partial_t A = -\nabla\Phi \Longrightarrow \mathbf{E} = -\nabla\boldsymbol{\Phi} - \mathbf{c}^{-1}\partial_{\mathbf{t}}\mathbf{A}$$

- Non-uniqueness of these representations requires to work in a restrictive class of potentials, called a *gauge*. A convenient choice of gauge here is the **Coulomb gauge**:

$$\nabla \cdot A = 0 \quad (\textit{Coulomb gauge condition}).$$

Potential formulation of (M) in Coulomb gauge

- In the Coulomb gauge, $\nabla \cdot A = 0$:

$$\nabla \cdot E = 4\pi\rho \Longrightarrow -\Delta\Phi - c^{-1}\partial_t \overbrace{(\nabla \cdot A)}^{0} = 4\pi\rho.$$

Then $\nabla \times B - c^{-1}\partial_t E = 4\pi c^{-1} j$ becomes:

$$\underbrace{\nabla \times (\nabla \times A)}_{-\Delta A} + c^{-2}\partial_{tt}^2 A = 4\pi c^{-1} j - c^{-1}\nabla(\partial_t \Phi)$$

- So Maxwell's equations (M) can be reformulated in terms of the potentials as (*the elliptic & hyperbolic PDEs*):

$$(\mathbf{M}) \begin{cases} -\Delta\Phi = 4\pi\rho \\ -\Delta A + c^{-2}\partial_{tt}^2 A = 4\pi c^{-1} j - c^{-1}\nabla(\partial_t \Phi) \\ \nabla \cdot A = 0 \end{cases}$$

Potential formulation of (D) in Coulomb gauge

- From the two decompositions $E = E_L + E_T$ and $E = -\nabla\Phi - c^{-1}\partial_t A$, we have

$$E_L = -\nabla\Phi \qquad \text{and} \qquad E_T = -c^{-1}\partial_t A.$$

- Therefore neglecting $c^{-1}\partial_t E_T$ in Maxwell's-Ampère's law is equivalent to neglecting $c^{-2}\partial_{tt}^2 A$ in its potential formulation (*that is the wave equation*).
- So Darwin's equations (D) can be reformulated in terms of the potentials as (*the 'elliptic' PDEs*):

$$(\mathbf{D}) \begin{cases} -\Delta\Phi = 4\pi\rho \\ -\Delta A = 4\pi c^{-1} j - c^{-1}\nabla(\partial_t \Phi) \\ \nabla\cdot A = 0 \end{cases}$$

The 'generalized' momentum variable

- Insert potential formulations of E and B in RV equation:

$$\partial_t f + v(\xi)\cdot\nabla_x f - \left[\nabla\Phi + c^{-1}\partial_t A - c^{-1}v(\xi)\times(\nabla\times A)\right]\cdot\nabla_\xi f = 0$$

- The characteristic system associated to this equation is:

$$\begin{cases} \dot{X}(t) = v\left(\Xi(t)\right) \\ \dot{\Xi}(t) = -\left[\nabla\Phi + c^{-1}\partial_t A - c^{-1}v\times(\nabla\times A)\right](t, X(t), \Xi(t)) \end{cases}$$

- Observe that $\dot{A}(t, X(t)) = \partial_t A + (v\cdot\nabla)A$. Then, it is more convenient to write the characteristic system in terms of:

$$\mathbf{p} := \xi + \mathbf{c}^{-1}\mathbf{A} \quad (\textit{the 'generalized' momentum variable}).$$

Then: $\dot{P}(t) = -\left[\nabla\Phi - c^{-1}\sum v^i\nabla A^i\right](t, X(t), \Xi(t))$

Vlasov equation in generalized phase space

- In the generalized phase space $\mathbb{R}^3_x \times \mathbb{R}^3_p$; $(p = \xi + c^{-1}A)$:

$$v(\xi) = \frac{\xi}{\sqrt{1 + c^{-2}|\xi|^2}} \longrightarrow v_A(t,x,p) = \frac{p - c^{-1}A}{\sqrt{1 + c^{-2}|p - c^{-1}A|^2}}$$

and the characteristic system for RV equation becomes:

$$\begin{cases} \dot{X}(t) = v_A\,(t, X(t), P(t)) \\ \dot{P}(t) = -\left[\nabla\Phi - c^{-1} v_A^i \nabla A^i\right](t, X(t), P(t)) \end{cases}$$

- The associated kinetic equation to this characteristic system is the Vlasov equation formulated in the generalized phase space $\mathbb{R}^3_x \times \mathbb{R}^3_p$ as: $f = f(t,x,p)$,

$$\textbf{(RV)} \quad \partial_t f + v_A \cdot \nabla_x f - \left[\nabla\Phi - c^{-1} v_A^i \nabla A^i\right] \cdot \nabla_p f = 0$$

RVD system in generalized phase space in Coulomb gauge

- The RVD system reformulated in the generalized phase space $\mathbb{R}^3_x \times \mathbb{R}^3_p$ and in terms of potentials in Coulomb gage is:

$$
\textbf{(RVD)} \quad \begin{cases} \partial_t f + v_A \cdot \nabla_x f - \left[\nabla\Phi - c^{-1} v_A^i \nabla A^i\right] \cdot \nabla_p f = 0 \\ -\Delta\Phi = 4\pi\rho \\ -\Delta A = 4\pi c^{-1} j_A - c^{-1}\nabla(\partial_t \Phi), \quad \nabla \cdot A = 0 \end{cases}
$$

where $\quad f = f(t,x,p), \quad v_A = \frac{p - c^{-1}A}{\sqrt{1+|p-c^{-1}A|^2}} \quad$ and

$$
\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,p)dp, \quad j_A(t,x) = \int_{\mathbb{R}^3} v_A f(t,x,p)dp.
$$

- **Remark:** If $c \to \infty$, (RVD) reduces to Vlasov-Poisson system:

$$
\partial_t f + \xi \cdot \nabla_x f - \nabla\Phi \cdot \nabla_\xi f = 0; \;\; -\Delta\Phi = 4\pi\rho.
$$

Previous results on RVD system

- **Benachour, Filbert, Laurencot, Sonnendrücker (2003):** Gobal existence of weak solutions to RVD system for small initial data.
- **Pallard (2006):** Global existence of weak solutions to RVD system for general initial data, and local existence and uniqueness of classical solutions for smooth data.
- **Seehafer (2008):** Global existence of classical solutions to RVD system for small initial data.
- **A., Illner, Sospedra-Alfonso (2012):** Global existence and uniqueness of classical solutions to RVD system for small initial data. (*The proof uses the formulation of the RVD system in terms of potentials in the generalized phase space; it is constructive, and generalizes the proof given by Rein (2007) for the Vlasov-Poisson system*).

Definition of weak solutions

If $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \geq 0$ and $T > 0$ are given, then $f = f(t,x,p)$ is a weak solution to (RVD) with initial datum f_0 if:

- $f \in C\left([0,T), L^1 \cap L^\infty(\mathbb{R}^6)\right)$, $f \geq 0$, $\|f(t)\|_{L^1(\mathbb{R}^6)} = \|f_0\|_{L^1(\mathbb{R}^6)} \quad \forall t \in [0,T)$.
- f induces some potentials (Φ, A) that satisfy Darwin's equations in a weak sense; (i.e., after multiplication of Eq. (D) by test functions $\varphi \in C_0^\infty\left([0,T) \times \mathbb{R}^3\right)$, and integrating by parts w.r.t. x).
- f satisfies Vlasov equation with $f|_{t=0} = f_0$, in a weak sense; (i.e., after multiplication of Eq. (RV) by test functions $\varphi \in C_0^\infty\left([0,T) \times \mathbb{R}^3 \times \mathbb{R}^3\right)$, and integrating by parts w.r.t. (t,x,p)).

The main result

- **Theorem.** *Let* $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \geq 0$ *with compact support in* $\mathbb{R}^3 \times \mathbb{R}^3$. *Let* $T > 0$ *and* f *be a weak solution to (RVD) on* $[0, T)$ *s.t.* $f(t)$ *has a compact support in* $\mathbb{R}^3 \times \mathbb{R}^3$ *for all* $t \in [0, T)$. *Then this solution is unique.*

- The proof uses techniques from optimal transport theory.
- It extends the proof given by Loeper (2006) for the uniqueness of weak solutions to the Vlasov-Poisson system under the assumption that the charge densities remain bounded at all times.
- One of the new difficulties arisen in the RVD system, as opposed to the Vlasov-Poisson system, is the presence of the vector potential for which new estimates are required.

Regularity of Darwin potentials, (assume now $c = 1$)

- Lemma
  Let f be given as in the theorem. Then f induces a pair of potentials (Φ, A) in $L^\infty\left([0,T), C_b^1(\mathbb{R}^3)\right)$ which solves Darwin system (D) in a weak sense. Moreover, A is a weak solution of

$$-\Delta A = 4\pi j_A + \nabla\left\{\nabla \cdot \int_{\mathbb{R}^3} j_A(y)\frac{dy}{|y-x|}\right\}, \quad \nabla \cdot A = 0.$$

  - **Proof:**
    • $\Phi(t,x) = \int_{\mathbb{R}^3} \rho(t,y)\frac{dy}{|y-x|}$ solves $-\Delta\Phi = 4\pi\rho$.
    • Insert $\Phi(t,x)$ into $-\Delta A = 4\pi j_A - \nabla(\partial_t \Phi)$ and use the charge conservation law, $\partial_t\rho + \nabla_x \cdot j_A = 0$, to obtain the above "Poisson's" equation for A.
    • Can show that $A(t)$ is a C_b^1 solution of the integral eqn: $A(t,x) = \frac{1}{2}\int_{\mathbb{R}^3}[\mathtt{id} + \omega\otimes\omega]\, j_A(t,y)\frac{dy}{|y-x|}$, $w := \frac{y-x}{|y-x|}$.

- Lemma

  Given a pair of Darwin potentials (Φ, A), consider the (linear!) Vlasov equation (RV) associated with these (fixed) potentials. Then the characteristic system
  $$\begin{cases} \dot{X}(t) = v_A(t, X(t), P(t)) \\ \dot{P}(t) = -\left[\nabla\Phi - v_A^i \nabla A^i\right](t, X(t), P(t)) \end{cases}$$
  admits a unique solution $Z(t,z) = (X(t,z), P(t,z))$ starting from $Z(0,z) = z = (x,p)$, and the function $f(t) = Z(t)_\# f_0$, defined by

  $$\int_{\mathbb{R}^6} \varphi(z) f(t,z) dz = \int_{\mathbb{R}^6} \varphi(Z(t,z)) f_0(z) dz$$

  is the unique solution of the (linear!) Vlasov eqn (RV) associated with these fixed potentials (Φ, A).

  - **Main ingredient of Proof:**
    $\nabla_x \cdot v_A - \nabla_p \cdot \left[\nabla\Phi - v_A^i \nabla A^i\right] = 0.$

Proof of the Uniqueness Theorem

Let f_1, f_2 be 2 weak solutions of (RVD) starting at f_0, as in the theorem.

- By the two lemmas 1 and 2,

$$f_1(t) = Z_i(t)_{\#} f_0;\; Z_1(t,z) = \left(X_1(t,z), P_1(t,z)\right),\; Z_1(0,z) = z,$$

$$\begin{cases} \dot{X}_1(t) = v_{A_1}\left(t, Z_1(t)\right) \\ \dot{P}_1(t) = -\left[\nabla\Phi_1 - v^i_{A_1}\nabla A^i_1\right](t, Z_1(t)), \end{cases}$$

$$-\Delta\Phi_1 = 4\pi\rho_1, \quad -\Delta A_1 = 4\pi j_{A_1} + \nabla\left\{\nabla\cdot\int_{\mathbb{R}^3} j_{A_1}(y)\frac{dy}{|y-x|}\right\}$$

- Consider $\quad Q(t) = \frac{1}{2}\int_{\mathbb{R}^6}\left|Z_1(t,z) - Z_2(t,z)\right|^2 f_0(z)dz.$
  Goal: To show that $Q(t) = 0 \;\; \forall t \in [0,T)$ knowing that $Q(0) = 0$, because $Z_1(0,z) = z = Z_2(0,z)$.

Steps of the proof

- (1) Compute $\dot{Q}(t)$
- (2) Estimate $\dot{Q}(t)$ in terms of $Q(t)$
- (3) Use Gronwall's inequality and $Q(0) = 0$, to show that $Q(t) = 0 \ \forall t \in [0, T)$.
- (4) Then conclude using the inequality:

$$W_2^2\left(f_1(t), f_2(t)\right) \le 2Q(t), \tag{1}$$

where $W_2\left(f_1(t), f_2(t)\right)$ is the L^2-*Wasserstein distance* between $f_1(t)$ and $f_2(t)$,

$$W_2^2\left(f_1(t), f_2(t)\right) = \inf\left\{ \int_{\mathbb{R}^{12}} |z-\bar{z}|^2 d\gamma(z, \bar{z});\ \ \gamma \in \Gamma(f_1(t), f_2(t))\right\}.$$

- Note that (1) follows easily by using $\gamma = (Z_1(t) \times Z_2(t))_{\#} f_0$ in the above inf problem for $W_2^2\left(f_1, f_2\right)$.

Sketch of Proof of the Theorem

- From the characteristic systems, we have:

$$\dot{Q}(t) = I_1(t) + I_2(t) + I_3(t) \quad \text{where}$$

$$I_1 = \int [X_1(t) - X_2(t)] \cdot [v_{A_1}(t, Z_1(t)) - v_{A_2}(t, Z_2(t))]\, f_0(z)dz$$

$$I_2 = \int [P_1(t) - P_2(t)] \cdot [\nabla \Phi_2(t, X_2(t)) - \nabla \Phi_1(t, X_1(t))]\, f_0(z)dz$$

$$I_3 = \int [P_1(t) - P_2(t)] \cdot \left[v_{A_1}^i \nabla A_1^i(t, Z_1) - v_{A_2}^i \nabla A_2^i(t, Z_2)\right] f_0(z)dz$$

- Next we estimate $I_1(t)$, $I_2(t)$ and $I_3(t)$ in terms of $Q(t)$.
- Note that $v_{A_j}(t,z) = \frac{p - A_j(t,x)}{\sqrt{1+|p-A_j(t,x)|^2}} = v(p - A_j(t,x))$, where $v(g) := \frac{g}{\sqrt{1+|g|^2}} \in C_b^1(\mathbb{R}^3)$ and $z = (x,p)$.

Estimate $I_1(t)$

- By the mean value theorem on $v(g)$, Cauchy-Schwarz inequality, and $A_j \in L^\infty\left([0,T), C_b^1(\mathbb{R}^3)\right)$ and $f_j \in C\left([0,T), L^1 \cap L^\infty(\mathbb{R}^6)\right)$, we have:

$$\begin{aligned} I_1 &\le C\left\{Q + Q^{\frac{1}{2}}\left(\int |A_1(t, X_2(t)) - A_2(t, X_2(t))|^2 f_0(z)\right)^{\frac{1}{2}}\right\} \\ &= C\left\{Q + Q^{\frac{1}{2}}\left(\int_{\mathbb{R}^3} |A_1(t,x) - A_2(t,x)|^2 \rho_2(t,x)dx\right)^{\frac{1}{2}}\right\} \end{aligned}$$

- **Claim 1.** (*To be proved by optimal transport techniques*). $\int_{\mathbb{R}^3} |A_1(t,x) - A_2(t,x)|^2 \rho_2(t,x)dx \le CW_2^2\left(f_1(t), f_2(t)\right)$.

  We then deduce: $\quad I_1(t) \le CQ(t)$.

Estimate $I_2(t)$

- Included in [Loeper, 2006] for the proof of uniqueness of weak solutions to the Valsov-Poisson system:

$$I_2(t) \leq CQ(t)\left[1 - \ln Q(t)\right]$$

provided $\|Z_1(t) - Z_2(t)\|_{L^\infty(\mathbb{R}^6)} \leq 1/e$, i.e., $Q(t) \leq 1/(2e)$.

The main ingredient needed to prove this estimate is:

- **Claim 2.** (*To be proved by optimal transport techniques*).

$$\int_{\mathbb{R}^3} |\nabla\Phi_2(t,x) - \nabla\Phi_1(t,x)|^2 \rho_1(t,x)dx \leq CW_2^2\left(f_1(t), f_2(t)\right)$$

Estimate $I_3(t)$

- $I_3(t)$ can be decomposed as:

$$I_3(t) = \int f_0(z)\,[P_1(t) - P_2(t)] \cdot \left[v^i_{A_1} - v^i_{A_2}\right] \nabla A^i_2(t)\,dz + \int f_0(z)\,[P_1(t) - P_2(t)] \cdot \left[\nabla A^i_1 - \nabla A^i_2\right] v^i_{A_1}(t)\,dz.$$

Using $|v_{A_j}| \leq 1$, $A_j \in L^\infty\left([0, T), C^1_b(\mathbb{R}^3)\right)$ and analogue Claim 2 for A_j, $I_3(t)$ can be estimated as in I_1 and I_2.

- Combining all these estimates, we have:

$$\dot{Q}(t) \leq CQ(t)\,[1 - \ln Q(t)]\,, \quad Q(0) = 0$$

which implies by Gronwall's type inequality that:

$$Q(t) = 0 \quad \forall t \in [0, T).$$

Proof of Claims 1 & 2

Claims 1 & 2 follow from the following lemma:

- **Lemma 3**. (*Estimates by optimal transport techniques*) Let $0 \leq f_1, f_2 \in L^1 \cap L^\infty(\mathbb{R}^6))$ with compact support s.t. $\|f_1\|_{L^1(\mathbb{R}^6)} = \|f_2\|_{L^1(\mathbb{R}^6)}$. Let Φ_i, A_i be their induced Darwin's potentials. Then
  - $\|\nabla\Phi_2 - \nabla\Phi_1\|_{L^2(\mathbb{R}^3)} \leq CW_2(f_1, f_2)$; (*this proves Claim 2*).
  - $\|\nabla A_2 - \nabla A_1\|_{L^2(\mathbb{R}^3)} \leq CW_2(f_1, f_2)$; (*this proves the analogue of Claim 2 for vector potentials needed for the estimate of* $I_3(t)$).
  - $\int_{\mathbb{R}^3} |A_1(x) - A_2(x)|^2 \rho_i(x) dx \leq CW_2^2(f_1, f_2) \quad \forall i = 1, 2$; (*this proves Claim 1*).

Ingredients of Optimal transport

- **Brenier (1987).** $W_2^2(f_1, f_2) = \int_{\mathbb{R}^6} |\mathcal{T}(z) - z|^2 f_1(z)$, where $\mathcal{T} = \nabla\psi : \mathrm{supp} f_1 \to \mathrm{supp} f_2$, Ψ is convex, $\mathcal{T}_{\#} f_1 = f_2$.
- **McCann (1997).** $\forall \theta \in [1, 2]$, $f_\theta := \mathcal{T}_{\theta\#} f_1$, where $\mathcal{T}_\theta := (2-\theta)\mathtt{id}_{\mathbb{R}^6} + (\theta - 1)\mathcal{T}$, is the geodesic connecting f_1 and f_2 w.r.t. the L^2-Wassertein distance W_2.
- **Benamou-Brenier (2000).** f_θ satisfies the continuity equation (in a weak sense): $\partial_\theta f_\theta(z) + \nabla_z \cdot [u_\theta(z) f_\theta(z)] = 0$; $u_\theta(\mathcal{T}_\theta(z)) = \partial_\theta \mathcal{T}_\theta(z)$. Then the L^2-Wasserstein distance is given by: $W_2^2(f_1, f_2) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |u_\theta(z)|^2 f_\theta(z) dz$.
- $\mathrm{supp} f_\theta$ is compact in $\mathbb{R}^3 \times \mathbb{R}^3$, uniformly w.r.t. $\theta \in [1, 2]$.
- $\|f_\theta\|_{L^\infty(\mathbb{R}^6)} \le \max\left\{\|f_1\|_{L^\infty(\mathbb{R}^6)}, \|f_2\|_{L^\infty(\mathbb{R}^6)}\right\}$.

Proof of Lemma 3 (estimate on $\nabla\Phi_j$)

- From $-\Delta\Phi_1 = 4\pi\rho_1$ and $-\Delta\Phi_2 = 4\pi\rho_2$, we have:

$$-\Delta(\Phi_2 - \Phi_1)(x) = 4\pi \int_{\mathbb{R}^3} (f_2 - f_1)(x,p)\,dp$$

- Multiply by $(\Phi_2 - \Phi_1)(x)$ and integrate over $x \in \mathbb{R}^3$:

$$\|\nabla\Phi_2 - \nabla\Phi_1\|^2_{L^2(\mathbb{R}^3)} = 4\pi \int_{\mathbb{R}^3\times\mathbb{R}^3} \underbrace{(f_2 - f_1)}_{\partial_\theta f_\theta}(\Phi_2 - \Phi_1)\,dxdp$$

- Use the continuity equation $\partial_\theta f_\theta(z) = -\nabla_z \cdot [u_\theta(z) f_\theta(z)]$, and integration by parts (*using that suppf_θ is compact*), and Cauchy-Schwarz inequality:

$$\|\nabla\Phi_2 - \nabla\Phi_1\|^2_{L^2} \leq 4\pi\|\rho_\theta\|^{\frac{1}{2}}_{L^\infty}\|\nabla\Phi_2 - \nabla\Phi_1\|_{L^2}\, W_2(f_1, f_2).$$

Proof of Lemma 3 (estimates on A_j)

The estimates on the vector potentials follow the same procedure, but some more linear algebra.

- Substract the 2 vector potential equations:

$$-\Delta(A_2 - A_1)(x) = 4\pi(j_{A_2} - j_{A_1})(x) + \nabla\left\{\nabla\cdot\int_{\mathbb{R}^3}(j_{A_2} - j_{A_1})(y)\frac{dy}{|y-x|}\right\}$$

  and $\nabla\cdot A_1 = \nabla\cdot A_2 = 0$ (*Coulomb gauge condition*).

- Multiply by $(A_2 - A_1)(x)$ and integrate over $x \in \mathbb{R}^3$. Integration by parts and $\nabla\cdot(A_2 - A_1) = 0$ yield:

$$\|\nabla A_2 - \nabla A_1\|^2_{L^2(\mathbb{R}^3)} = 4\pi\int_{\mathbb{R}^3}(A_2 - A_1)\cdot(j_{A_2} - j_{A_1})dx$$

Proof of Lemma 3 (estimates on A_j) ...

- $j_{A_2} - j_{A_1} = \int_{\mathbb{R}^3} f_2(v_{A_2} - v_{A_1})dp + \int_{\mathbb{R}^3} v_{A_1}(f_2 - f_1)dp$:

$$\|\nabla A_2 - \nabla A_1\|_{L^2} - 4\pi \int (A_2 - A_1) \cdot (v_{A_2} - v_{A_1}) f_2 dpdx$$
$$= 4\pi \int v_{A_1} \cdot (A_2 - A_1) \underbrace{(f_2 - f_1)}_{\partial_\theta f_\theta} dpdx.$$

- Use the continuity equation $\partial_\theta f_\theta(z) = -\nabla_z \cdot [u_\theta(z) f_\theta(z)]$, integration by parts, Cauchy-Schwarz inequality, $A_j \in C_b^1(\mathbb{R}^3)$, $v_{A_j} \leq 1$, and the Poincaré's inequality $\|A_2 - A_1\|_{L^2(\operatorname{supp} f_\theta)} \leq \|\nabla A_2 - \nabla A_1\|_{L^2(\operatorname{supp} f_\theta)}$:

$$RHS \leq C\|\rho_\theta\|_{L^\infty}^{\frac{1}{2}} \|\nabla A_2 - \nabla A_1\|_{L^2} W_2(f_1, f_2).$$

Proof of Lemma 3 (estimates on A_j) ...

- **Linear algebra.** Since $v(g) := \frac{g}{\sqrt{1+|g|^2}} \in C_b^1(\mathbb{R}^3)$, $Dv(g)$ is a positive definite matrix with $\det Dv(g) = (1+|g|^2)^{-5/2}$. So writing $v_{A_j} = v(g_{A_j})$, $g_{A_j} := p - A_j$, and using the mean value theorem on $v(g)$, we have:

$$
\begin{aligned}
-(v_{A_2} - v_{A_1}) \cdot (A_2 - A_1) &= Dv(g)(A_2 - A_1) \cdot (A_2 - A_1) \\
&\geq \lambda |A_2 - A_1|^2,
\end{aligned}
$$

  for some $\lambda > 0$ (uniformly in (x, p)), so that

$$
LHS \geq \|\nabla A_2 - \nabla A_1\|^2_{L^2(\mathbb{R}^3)} + 4\pi\lambda \int_{\mathbb{R}^3} |A_1(x) - A_2(x)|^2 \rho_2(x) dx
$$

- Insert these 2 estimates into $LHS = RHS$ to conclude.

That's it!

Thank you!